Subgraph Conditions for Hamiltonian Properties of Graphs

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The research presented in this dissertation was funded by and carried out at the group of Discrete Mathematics and Mathematical Programming (DMMP), Department of Applied Mathematics, Faculty of Electrical Engineering, Mathematics and Computer Science of the University of Twente, the Netherlands.

The financial support from University of Twente for this research work is gratefully acknowledged.

The thesis was typeset in LaTeX by the author and printed by Gildeprint Drukkerijen, Enschede, the Netherlands.
http://www.gildeprint.nl

ISBN 978-90-365-3403-1

SUBGRAPH CONDITIONS FOR HAMILTONIAN PROPERTIES OF GRAPHS

PROEFSCHRIFT

ter verkrijging van
de graad van doctor aan de Universiteit Twente,
op gezag van de rector magnificus,
prof. dr. H. Brinksma,
volgens besluit van het College voor Promoties
in het openbaar te verdedigen
op donderdag 20 september 2012 om 16:45 uur

door

Binlong Li

geboren op 2 augustus 1983
te Hebei, China

Dit proefschrift is goedgekeurd door de promotoren

prof. dr. ir. H.J. Broersma en prof. dr. S. Zhang

Preface

This thesis consists of an introductory chapter (Chapter 1) followed by eight research chapters (Chapters 2–9), each of which is written as a self-contained journal paper, except that all references are gathered at the end of the thesis. These eight chapters are based on the eight papers that are listed below and have been submitted to journals for publication. Chapters 2, 3 and 6 are mainly based on research that was done while the author was working as a PhD student at Northwestern Polytechnical University in Xi'an, China; the other chapters are mainly based on research of the author at the University of Twente. The paper that forms the basis for Chapter 7 has recently been published in Discrete Mathematics, and the paper underlying Chapter 6 has been accepted for SIAM journal on Discrete Mathematics. The other papers are in different stages of the refereeing process. All chapters deal with results in which certain subgraph conditions on graphs imply that these graphs have structural properties that are somehow related to the existence of Hamilton cycles. This explains the title of the thesis. Since the thesis has been written as a collection of more or less independent papers, the reader will find a certain amount of repetition of relevant concepts, definitions and background. The author apologizes for any inconvenience.

Papers underlying this thesis

[1] B. Li and S. Zhang, Heavy subgraph conditions for longest cycle to be heavy in graphs, preprint. (Chapter 2)

[2] B. Li and S. Zhang, On traceability of claw-o_{-1}-heavy graphs, preprint. (Chapter 3)

[3] B. Li, H.J. Broersma and S. Zhang, Forbidden subgraph pairs for traceability of block-chains, preprint. (Chapter 4)

[4] B. Li, H.J. Broersma and S. Zhang, Heavy subgraph pairs for traceability of block-chains, preprint. (Chapter 5)

[5] B. Li, Z. Ryjáček, Y. Wang and S. Zhang, Pairs of heavy subgraphs for Hamiltonicity of 2-connected graphs, SIAM J. Disc. Math., to appear. (Chapter 6)

[6] B. Li, H.J. Broersma and S. Zhang, Pairs of forbidden induced subgraphs for homogeneously traceable graphs, Disc. Math., 312 (2012), 2800–2818. (Chapter 7)

[7] B. Li, B. Ning, H.J. Broersma and S. Zhang, Characterizing heavy subgraph pairs for pancyclicity, preprint. (Chapter 8)

[8] B. Li, H.J. Broersma and S. Zhang, Heft index, separable degree and path partition, preprint. (Chapter 9)

Contents

Chapter 1

Introduction

1.1 Basic terminology and background

For terminology and notation not defined here, we use Bondy and Murty [8]. We consider finite, undirected, simple graphs only.

A graph G is *hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle containing all vertices of G. The term refers to Sir William Rowan Hamilton who invented a game in the 1850s in which a player has to produce a Hamilton cycle in a dodecahedron after another player has prescribed five consecutive vertices of it. We omit the details.

Checking whether a given graph G is hamiltonian or not is a notorious *NP-complete* decision problem, and is a special case of the *Traveling Salesman Problem* that attracted a lot of attention (See, e.g., [1]). Since this thesis deals with structural conditions for hamiltonian properties and not with algorithmic questions, we will not elaborate on the complexity issues involved, but we refer the interested reader to the vast literature that can easily be found on the internet.

In contrast to the problem of deciding whether a given graph is *eulerian*, i.e., contains a trail that traverses every edge of the graph exactly once, no nice *characterization*, i.e., a necessary and sufficient condition, is known for the existence of a Hamilton cycle in a graph, in the sense of being useful in deciding the (non)existence of such a cycle without too much effort. Since the early 1950s this has been the motivation for considering necessary conditions and sufficient conditions separately, with a strong emphasis on sufficient conditions. The results of this thesis also mainly deal with sufficient conditions, although we sometimes add a mild necessary condition to the graph classes we consider in order to obtain stronger results. We will come back to this later.

We will shortly describe two types of sufficient conditions for the existence of a Hamilton cycle that have been popular research areas for a considerable time, namely *degree conditions* and *forbidden subgraph conditions*. Before we do so, we need to introduce some additional terminology.

Let G be a graph. For a vertex $v \in V(G)$ and a subgraph H of G, we use $N_H(v)$ to denote the set, and $d_H(v)$ to denote the number, of neighbors of v in H, respectively. We call $d_H(v)$ the *degree* of v in H. For $x, y \in V(G)$, an (x, y)*-path* is a path P connecting x and y; the vertex x will be called the *origin* and y the *terminus* of P. If $x, y \in V(H)$, the *distance* between x and y in H, denoted $d_H(x, y)$, is the length of a shortest (x, y)-path in H. If there are no (x, y)-paths in H, then we define $d_H(x, y) = \infty$. When no confusion can occur, we will denote $N_G(v)$, $d_G(v)$ and $d_G(x, y)$ by $N(v)$, $d(v)$ and $d(x, y)$, respectively.

The earliest degree condition for a graph to be hamiltonian was given by Dirac [20] in 1952. It states that a graph G on $n \geq 3$ vertices is hamiltonian if every vertex of G has degree at least $n/2$. Dirac's Theorem has been generalized in several ways and directions. For later reference we present the following degree sum condition given by Ore [30] in 1960. We will present generalizations of this result and its counterpart for other hamiltonian properties in the thesis.

Theorem 1.1 (Ore [30])**.** *Let G be a graph on $n \geq 3$ vertices. If for every two nonadjacent vertices $u, v \in V(G)$, $d(u) + d(v) \geq n$, then G is hamiltonian.*

These early degree conditions and many of its successors have a serious drawback. Although they are best possible in the sense that we cannot replace n by $n-1$ in the above result, the graphs satisfying the conditions are very close to complete graphs and therefore almost trivially hamiltonian. For example, the graphs satisfying the condition in Ore's Theorem have at least roughly $n^2/8$ edges. In fact, they are close to complete graphs in the following sense: one can add edges one by one between nonadjacent vertices in such a way that the new graph is hamiltonian if and only if the previous graph is hamiltonian, until a complete graph has been obtained. This follows from a well-known *closure* result of Bondy and Chvátal [7]. We omit the details.

The above degree conditions are sometimes referred to as numerical conditions or global conditions, for obvious reasons, and seem to be too strong for guaranteeing hamiltonicity, in the sense that they imply much more on the structure of the graphs that satisfy these conditions. This might have been a reason for researchers to consider structural instead of numerical conditions, and local instead of global conditions. One option is to look at local structures of the graph and impose certain conditions there.

We now turn to subgraph conditions and the relevant terminology and notation. Let G be a graph. If a subgraph G' of G contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G (or a subgraph of G induced by $V(G')$). For a given graph H, we say that G is *H-free* if G does not contain an induced subgraph isomorphic to H. For a family $\mathcal{H}$ of graphs, G is called *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$. If G is H-free, then H is called a *forbidden subgraph* of G. Note that forbidding H as an induced subgraph puts less restrictions on the graph G than forbidding H as a subgraph: in the former case H is allowed as a subgraph of G if G contains at least two adjacent vertices that are nonadjacent in H. Also note that the conditions on the graph G become weaker if the forbidden subgraph gets larger, in the following sense: if H_1 is an induced subgraph of H_2, then G being H_1-free implies that G is H_2-free, but not vice versa. So the larger the forbidden subgraphs, the richer the class of graphs under consideration, in the above sense.

We will now describe some special graphs and graph classes that play a key role as forbidden induced subgraphs in the sequel.

The graph $K_{1,3}$ is called a *claw* . The vertex with degree 3 is called the *center* , and the other vertices are the *end-vertices* of the claw. Claw-free graphs have been a very popular field of study, not only in the context of hamiltonian properties. One reason is that the very natural class of *line graphs* turns out to be a subclass of the class of claw-free graphs. So it is a rich class in the sense that for every graph $G = (V, E)$ we can obtain a (claw-free) line graph $L(G)$, with vertices of $L(G)$ corresponding to the edges of E, and with two vertices adjacent in $L(G)$ if and only if the corresponding edges of G share exactly one vertex in G. It is an easy exercise to show that line graphs cannot contain a claw as an induced subgraph. In fact, line graphs can be characterized by a set of nine forbidden subgraphs, one being the claw. We will not elaborate on this in the thesis. Forbidding the claw does not help for hamiltonicity, i.e., not every claw-free graph is hamiltonian. There are examples of 3-connected nonhamiltonian claw-free (even line) graphs, but it is a long-standing conjecture that all 4-connected claw-free graphs are hamiltonian. It is interesting to note that the lower bound on the degrees in Dirac's Theorem can be lowered to roughly $n/3$ in case of claw-free graphs and something similar holds for the bound in Ore's Theorem. Natural questions to consider here are: is there a single (connected) graph H such that every (2-connected) H-free graph is hamiltonian? Is there a (connected) graph H such that every (2-connected) claw-free H-free graph is hamiltonian? Can we characterize all such graphs or pairs of graphs for this and other hamiltonian properties? This is the motivation for the results of this thesis.

Let P_i $(i \geq 1)$ be the path on i vertices, and C_i $(i \geq 3)$ be the cycle on i vertices. We use Z_i $(i \geq 1)$ to denote the graph obtained by identifying a vertex of a C_3 with an end vertex of a P_{i+1}, $B_{i,j}$ $(i, j \geq 1)$ to denote the graph obtained by identifying two vertices of a C_3 with the origins of a P_{i+1} and a P_{j+1}, respectively, and $N_{i,j,k}$ $(i, j, k \geq 1)$ to denote the graph obtained by identifying the three vertices of a C_3 with the origins of a P_{i+1}, P_{j+1} and P_{k+1}, respectively. In particular, we let $B = B_{1,1}$ (this graph is sometimes called a *bull*), $W = B_{1,2}$ (this graph is sometimes called a *wounded*) and $N = N_{1,1,1}$ (this graph is sometimes called a *net*) (see Figure 1.1).

Forbidden subgraph conditions for hamiltonicity have been known since the early 1980s, but Bedrossian was the first to study the characterization of all pairs of forbidden graphs for hamiltonian properties in his PhD thesis of 1991 [3].

Before we state one of his results, we first note that forbidding K_1 is absurd because we always assume a graph has a nonempty vertex set. Moreover, we note that a K_2-free graph is an empty graph (contains no edges), so it is trivially nonhamiltonian. In the following and throughout the thesis, we therefore assume that all the forbidden subgraphs we will consider have at least three vertices. We also restrict our attention to connected forbidden subgraphs, since we want to look at local conditions, in the sense that the

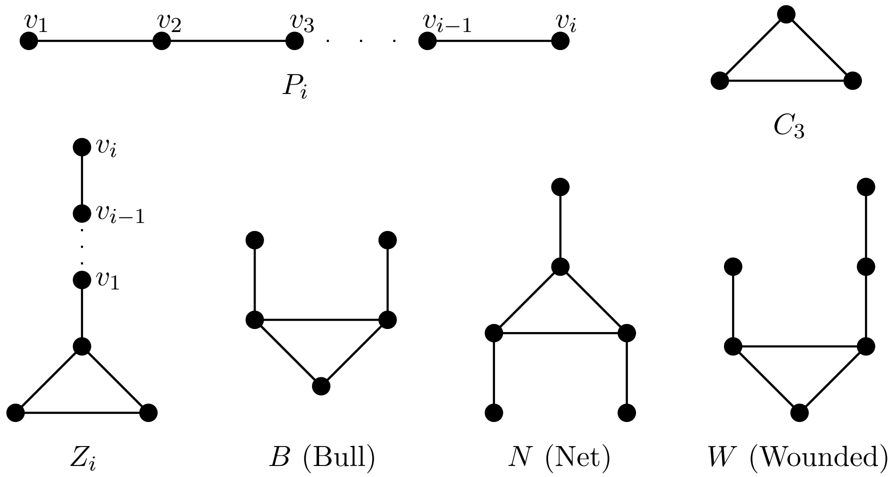


Figure 1.1: Graphs P_i, C_3, Z_i, B, N and W

vertices of the concerning subgraph have a distance not so far in the graphs. Finally, we note that every component of a P_3-free graph is a complete graph. Hence a connected P_3-free graph on at least 3 vertices is trivially hamiltonian, and it is in fact easy to show that P_3 is the only connected graph H such that every connected H-free graph on at least 3 vertices is hamiltonian. The next result of Bedrossian deals with pairs of forbidden subgraphs, excluding P_3.

Theorem 1.2 (Bedrossian [3]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, P_6, C_3, Z_1, Z_2, B, N or W.*

Important to note here is that the claw is always one of the forbidden subgraphs, a phenomenon that we often encounter(ed) in similar results. Also recall that a P_4-free graph is P_5-free, etc., so the relevant graphs for S are in fact P_6, N and W; all the other listed graphs are induced subgraphs of P_6, N or W.

Motivated by Bedrossian's results many papers have appeared in which similar results have been obtained for other graph properties. We will come back to this later, and we will present a newly obtained result in the thesis for the property of being homogeneously traceable, to be defined later.

One of the main objects of the thesis, however, is to combine the two types of conditions, i.e., to restrict the degree conditions to certain subgraphs. Why should we be interested in doing so? Recall that the degree conditions had the drawback that they impose such strong conditions on the graphs that they are not far from complete graphs, in the sense described earlier. Early subgraph conditions have a similar drawback, especially if the forbidden subgraphs are small. As an example, it is an easy exercise to show that a connected $K_{1,3}$-free and Z_1-free graph is either a path, a cycle, or a complete graph minus the edges of a matching. We omit the details. Combining degree conditions and subgraph conditions by relaxing the degree conditions to hold for certain nonadjacent pairs of vertices in certain induced subgraphs instead of all nonadjacent pairs could clearly lead to common generalizations: if the degree condition holds for every nonadjacent pair, it obviously holds for certain nonadjacent pairs; allowing a certain subgraph as an induced subgraph under some condition is obviously weaker than forbidding the same subgraph.

Before we present the results of the thesis, we need a few more definitions. We first turn to a type of conditions for hamiltonian properties that we will generally address as *heavy subgraph conditions* .

Let G be a graph on n vertices, and let G' be an induced subgraph of G. We say that G' is *heavy* in G if there are two nonadjacent vertices in $V(G')$ with degree sum at least n in G. For a given fixed graph H, the graph G is called *H-heavy* if every induced subgraph of G isomorphic to H is heavy. For a family $\mathcal{H}$ of graphs, G is called *$\mathcal{H}$-heavy* if G is H-heavy for every $H \in \mathcal{H}$.

For hamiltonicity we obtained the following counterpart of Bedrossian's Theorem. The proof of this theorem can be found in Chapter 6 of this thesis.

Theorem 1.3. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, C_3, Z_1, Z_2, B, N or W.*

Comparing the two theorems, we firstly note that the claw $K_{1,3}$ is always one of the heavy pairs. Secondly, note that P_6 is the only graph that appears in the list of Bedrossian's Theorem but is missing here. Chapter 6 contains examples showing that P_6 has to be excluded in the above theorem.

In the thesis we will consider a number of other hamiltonian properties, i.e., properties that are similar to being hamiltonian but either weaker (implied by being hamiltonian) or stronger (implying hamiltonicity). We present some additional definitions first.

We begin with some weaker hamiltonian properties, the first of which is well-studied, but the second of which is not so well-known.

A graph G is said to be *traceable* if it contains a *Hamilton path* , i.e., a path containing all the vertices of G; it is called *homogeneously traceable* if for every vertex x of G, it contains a Hamilton path starting from x.

We also considered the following stronger hamiltonian properties.

A graph G is said to be *Hamilton-connected* if for every two distinct vertices x and y of G, it contains a Hamilton path connecting x and y; it is called *pancyclic* if it contains a cycle of length k for all k with $3 \leq k \leq n$, where $n = |V(G)|$.

For some properties, e.g., traceability and pancyclicity, we would like to consider a slightly weaker or stronger degree condition than the heavy subgraph condition we introduced before. We shall give the reasons for this later in this chapter. Here we introduce the additional related terminology.

Let G be a graph on n vertices, let G' be an induced subgraph of G, and let k be an integer. We say that G' is *o_k-heavy* in G if there are two nonadjacent vertices in $V(G')$ with degree sum at least $n+k$ in G. Here the o refers to the degree condition in Ore's Theorem, while the $_k$ refers to adding or subtracting a small constant in the degree condition. For a given fixed graph H, the graph G is called *H-o_k-heavy* if every induced subgraph of G isomorphic to H is o_k-heavy. For a family $\mathcal{H}$ of graphs, G is called *$\mathcal{H}$-o_k-heavy* if G is H-o_k-heavy for every $H \in \mathcal{H}$. Thus for $k = 0$, an H-o_0-heavy ($\mathcal{H}$-o_0-heavy) graph is an H-heavy ($\mathcal{H}$-heavy) graph.

Note that an H-free graph is also H-o_k-heavy; more generally, if $k \leq \ell$, then an H-o_ℓ-heavy graph is also H-o_k-heavy; if H_1 is an induced subgraph of H_2, then an H_1-free (H_1-heavy, H_1-o_k-heavy) graph is also H_2-free (H_2-heavy, H_2-o_k-heavy); and for a complete graph K_r, saying that a graph is K_r-free is equivalent to saying that it is K_r-heavy (K_r-o_k-heavy).

For the same reasons as before with forbidden subgraphs, when we say that a graph is H-heavy (H-o_k-heavy), we always assume by default that H has at least three vertices and that it is connected.

1.2 Main results of the thesis

The thesis contains a variety of results on subgraph conditions for hamiltonian properties of graphs. The general questions that have been addressed are: for which graph S, or for which pair of graphs R, S, does the following hold: every graph (restricted to a certain class of graphs, avoiding more or less trivial counterexamples) that is S-free (S-heavy, S-o_k-heavy) or $\{R, S\}$-free ($\{R, S\}$-heavy, $\{R, S\}$-o_k-heavy) has a certain hamiltonian property. For some properties, forbidden subgraph conditions were already established by other researchers; in that case, we present and prove the corresponding heavy subgraph counterparts; for other properties, we give both forbidden and heavy subgraph conditions for a graph to have the required property.

Let $\mathcal{P}$ be a property of graphs (like hamiltonicity, traceability, and so on). If apart from some trivial exceptions, a graph with property $\mathcal{P}$ must have (vertex) connectivity at least k, then we say that being k-connected is a *necessary connectivity condition* for property $\mathcal{P}$ (or that k is the *necessary connectivity* for property $\mathcal{P}$). For instance, every hamiltonian graph is 2-connected. Therefore being 2-connected is a necessary connectivity condition for the property hamiltonicity. When we consider the property $\mathcal{P}$, we only consider graphs that satisfy the necessary connectivity condition.

Another remark concerns the degree conditions we impose on certain nonadjacent vertices (for some types of heavy subgraph conditions). When we consider a hamiltonian property $\mathcal{P}$, it is always easy to construct a graph with a large minimum degree that does not satisfy the property $\mathcal{P}$. For instance, the complete bipartite graph $K_{(n-2)/2,(n+2)/2}$ on n vertices (with n even) is not traceable, and every induced subgraph of it (other than K_1 and K_2) is o_{-2}-heavy. On the other hand, a counterpart of Ore's Theorem shows that every graph on n vertices in which every pair of nonadjacent vertices has degree sum at least $n-1$, is traceable. This is the reason for considering o_{-1}-heavy subgraph conditions for traceability, where the subscript $n-1$ is called the *necessary degree sum* for traceability. This of course does not mean that a large degree sum of nonadjacent pairs of vertices is a necessary condition for traceability: a long path is a traceable graph but has maximum degree 2. Similarly, noting that $K_{(n-1)/2,(n+1)/2}$ is not hamiltonian and not homogeneously traceable, and $K_{n/2,n/2}$ is not pancyclic, the necessary degree sum for hamiltonicity and homogeneous traceability is n and the necessary degree sum for pancyclicity is $n+1$. Thus, for the hamiltonian property with necessary degree sum $n+k$, we always consider o_k-heavy subgraph conditions instead of heavy subgraph conditions.

Recall that if a connected graph is P_3-free, then it is a complete graph, and it satisfies all the above properties (with the corresponding necessary connectivity). In many cases, P_3 is the only single connected graph S such that every S-free graph (with the corresponding necessary connectivity) satisfies the given property. In fact, for o_k-heavy subgraph conditions, where $n+k$ is the corresponding necessary degree sum, this is also true. This will be proved in the respective chapters. In the remainder of this introduction, we will consider the more interesting cases involving pairs of subgraphs, except for the next subsection on longest cycles. So when we consider a pair of forbidden subgraphs (and also o_k-heavy subgraphs) in the sequel, we will always exclude P_3 as one of the members of the pairs.

A. A result on longest cycles

In Chapter 2 of the thesis we consider sufficient conditions for a property on longest cycles of a graph. We first introduce some additional terminology.

Let G be a graph on n vertices. A vertex v is called a *heavy vertex* of G if $d(v) \geq n/2$, and a cycle C is called a *heavy cycle* of G if C contains all the heavy vertices of G. From results by Bollobás and Brightwell [6] or Shi [33], one can easily deduce that every 2-connected graph has a heavy cycle. This result generalizes Dirac's Theorem, because if every vertex has degree at least $n/2$, the heavy cycle is a Hamilton cycle.

In general, a longest cycle of a graph need not necessarily be a heavy cycle. In Chapter 2 we consider the property that 'every longest cycle is a heavy cycle' in graphs. This property is clearly weaker than hamiltonicity.

Since a separable graph can have no cycles containing internal vertices of all its blocks, we only consider 2-connected graphs, although 2-connectivity is not a necessary condition for the property that every longest cycle is a heavy cycle. Since the definition of a heavy cycle involves the concept of a heavy vertex, we consider (forbidden subgraph conditions and) heavy subgraph conditions for this property, although n is not the necessary degree sum for this property. In both respects, Chapter 2 differs from the other chapters.

With respect to a single forbidden (or heavy) subgraph condition for the property that every longest cycle is a heavy cycle, for 2-connected graphs we obtained the following result.

Theorem 1.4. *Let S be a fixed connected graph and let G be an arbitrary 2-connected graph. Then G being S-free (or S-heavy) implies that every longest cycle of G is a heavy cycle, if and only if $S = P_3$, $K_{1,3}$ or $K_{1,4}$.*

Since the single forbidden (or heavy) subgraph is not always P_3, we expect that a characterization of all the pairs of forbidden (or heavy) subgraphs for this property will be very complicated. In this thesis, we do not consider pairs of forbidden (or heavy) subgraphs for this property.

B. Results on traceability

With respect to forbidden subgraph conditions for traceability of connected graphs, the following result was established in 1997.

Theorem 1.5 (Faudree and Gould [24]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, C_3, Z_1, B or N.*

It is a bit disappointing that one needs to forbid almost the same graphs as for hamiltonicity, i.e., a claw combined with any of the induced subgraphs of the net N, whereas traceability is a weaker property. The counterpart on heavy subgraphs does also indicate that traceability requires a strong hypothesis. Without any additional assumptions on the structure of the graph G, for o_{-1}-heavy subgraph conditions, perhaps surprisingly there exists only one pair for the property of traceability. The following result will be proved in Chapter 3.

Theorem 1.6. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$-o_{-1}-heavy implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$.*

Recall that C_3-o_{-1}-heavy is in fact equivalent to triangle-free. In order to obtain better results, it was observed that many graphs that were used to prove the 'only-if' part of the above theorem were almost trivially nontraceable, in the sense that they contain at least three end blocks. To exclude such graphs, we turned to block-chains, as defined below.

C. More results on traceability

A *block-chain* is a graph whose block graph is a path, i.e., it is either a P_1, a P_2, or a 2-connected graph, or a graph with at least one cut-vertex and exactly two end blocks. Note that every traceable graph is necessarily a block-chain, but that the reverse does not hold in general. Also note that it is easy to check by a polynomial algorithm whether a given graph is a block-chain. For the forbidden or heavy subgraph conditions for a block-chain to be traceable, we obtained the following results, the proofs of which can be found in Chapters 4 and 5, respectively. In the next theorem, the graph $N_{1,1,3}$ is the graph illustrated in Figure 1.2.

Theorem 1.7. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a block-chain. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,3}$, or $R = K_{1,4}$ and $S = P_4$.*

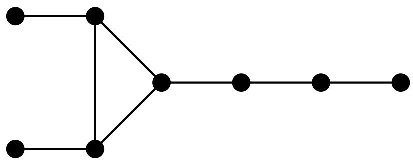


Figure 1.2: Graph $N_{1,1,3}$

It is interesting to note that one of the pairs does not include the claw, in contrast to all existing characterizations of pairs of forbidden subgraphs for hamiltonian properties we encountered.

Theorem 1.8. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a block-chain. Then G being $\{R, S\}$-o_{-1}-heavy implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of N or W.*

Note that we can relax C_3-o_{-1}-heavy (triangle-free) to a condition on much larger subgraphs by turning to block-chains.

D. Results on hamiltonicity

Bedrossian [3] studied forbidden subgraph conditions for a 2-connected graph to be hamiltonian. Recall that he characterized all pairs of forbidden subgraphs for hamiltonicity.

Theorem 1.9 (Bedrossian [3])**.** *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, P_6, C_3, Z_1, Z_2, B, N or W.*

For hamiltonicity of 2-connected graphs, we obtained the following counterpart on heavy subgraph pairs. The proof can be found in Chapter 6.

Theorem 1.10. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, C_3, Z_1, Z_2, B, N or W.*

As noted before, there is only one forbidden subgraph pair $\{K_{1,3}, P_6\}$ that is not a heavy pair for hamiltonicity.

E. Results on homogeneously traceable graphs

Note that a hamiltonian graph is homogeneously traceable, and that a homogeneously traceable graph is traceable, but not vice versa, so this condition is somewhere strictly between hamiltonicity and traceability. Also note that a homogeneously traceable graph is necessarily 2-connected. As far as we are aware, this property has not been studied before in the context of forbidden subgraphs, so we do not know of any existing forbidden subgraph results for homogeneously traceable graphs. We prove the following characterization of all such pairs in Chapter 7. The crucial graphs for this result are depicted in Figure 1.3.

Theorem 1.11. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is homogeneously traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $B_{1,4}$, $B_{2,3}$ or $N_{1,1,3}$.*

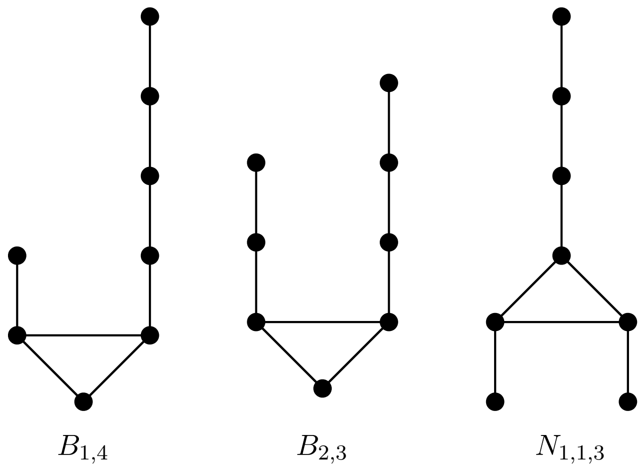


Figure 1.3: The graphs $B_{1,4}$, $B_{2,3}$ and $N_{1,1,3}$

For heavy subgraph conditions, we get the following counterpart of the above theorem in Chapter 7.

Theorem 1.12. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-heavy implies G is homogeneously traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, C_3, Z_1, Z_2, B, N or W.*

One may note that the heavy subgraph pairs in the above theorem are exactly the same as in the theorem for hamiltonicity. In fact, the 'if' part of the theorem can be deduced by the fact that every hamiltonian graph is homogeneously traceable. Some families of graphs that are not homogeneously traceable and that we need for the proof of the 'only-if' part are shown in Chapter 7.

F. Results on pancyclicity

In Bedrossian's PhD thesis, he also studied forbidden subgraph conditions for pancyclicity of 2-connected graphs and obtained the following result.

Theorem 1.13 (Bedrossian [3]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being*

$\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, Z_1 or Z_2.

With respect to o_1-heavy subgraph conditions for pancyclicity, we extended Bedrossian's result and obtained the following counterpart, the proof of which can be found in Chapter 8.

Theorem 1.14. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-o_1-heavy implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, Z_1 or Z_2.*

Note that exactly the same graphs appear in both results.

G. Results on path partition optimality

A *path partition* of a graph G is the union of some pairwise vertex-disjoint paths such that every vertex of G is contained in one of the paths. If G is a nonhamiltonian graph, then the *path partition number* of G, denoted by $\pi(G)$, is the minimum number of paths in a path partition of G; if G is hamiltonian, then we define $\pi(G) = 0$. Alternatively, $\pi(G)$ is the minimum number of edges we have to add to G to turn it into a hamiltonian graph, except for degenerate cases. Note that $\pi(K_1) = \pi(K_2) = 1$ and $\pi(2K_1) = 2$.

The *separable degree* of a graph G, denoted by $\sigma(G)$, is defined as the minimum number of edges one has to add to G to turn it into a 2-connected graph, again except for degenerate cases. We define $\sigma(K_1) = \sigma(K_2) = 1$ and $\sigma(2K_1) = 2$.

It is not difficult to see that for every graph G, $\pi(G) \geq \sigma(G)$. We call a graph *path partition optimal* if its path partition number is equal to its separable degree. In the final chapter of this thesis, we consider the path partition optimality of graphs.

With respect to forbidden subgraph conditions for a graph to be path partition optimal, we obtained the following result.

Theorem 1.15. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a graph. Then G being $\{R, S\}$-free implies G is path partition optimal if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$, P_4, Z_1, B or N.*

Before stating the counterpart of the above theorem for heavy subgraphs, we introduce some additional terminology and notation.

Let G be a graph and let G' be an induced subgraph of G. We define the *heft* of G' in G, denoted by $h_G(G')$ (or briefly, $h(G')$), as the maximum degree sum of two nonadjacent vertices in $V(G')$. If G' is a clique, then we define $h(G') = 0$. For a given graph H, the *H-heft index* of G, denoted by $\eta_H(G)$, is the minimum heft of an induced subgraph of G isomorphic to H. If G is H-free, then we define $\eta_H(G) = \infty$. Note that if H_1 is an induced subgraph of H_2, then $\eta_{H_1}(G) \leq \eta_{H_2}(G)$.

We use $n(G)$ to denote the order of G. Thus, a graph G with $\eta_H(G) \geq n(G) + k$ is an H-o_k-heavy graph.

With respect to heavy subgraph conditions for this property, we obtained the following result.

Theorem 1.16. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a graph. Then $\eta_R(G) \geq n(G) - \sigma(G)$ and $\eta_S(G) \geq n(G) - \sigma(G)$ implies G is path partition optimal, if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$, P_4, Z_1, B or N.*

1.3 Closure theory

In this section we use the terms claw-free, claw-heavy and claw-o_k-heavy instead of $K_{1,3}$-free, $K_{1,3}$-heavy and $K_{1,3}$-o_k-heavy, respectively.

Apart from the Bondy-Chvátal closure theorem based on degree sums of nonadjacent vertices that we mentioned before, there are two other types of closure theories that are closely related to the topic of the thesis. One was proposed by Ryjáček, in the context of his research on hamiltonicity of claw-free graphs; the other was proposed by Čada, for research on hamiltonicity of claw-heavy graphs. We distinguish them with a prefix or superscript r or c, respectively, in the following notations.

To study the hamiltonicity of claw-free graphs, in particular to show that the conjectures on hamiltonicity of 4-connected claw-free graphs and of 4-connected line graphs are equivalent, Ryjáček developed his closure theory, as follows.

Let G be a claw-free graph and let x be a vertex of G. We call x an *r-eligible* vertex if $N(x)$ induces a connected graph in G but not a complete graph. The *completion* of G at x, denoted by G'_x, is the graph obtained from

G by adding all missing edges uv with $u, v \in N(x)$. The following statement was proved by Ryjáček, where $c(G)$ is the length of a longest cycle of G.

Theorem 1.17 (Ryjáček [32]). *Let G be a claw-free graph, and let x be an r-eligible vertex of G. Then*

(1) *the graph G'_x is claw-free; and*

(2) $c(G'_x) = c(G)$.

Let G be a claw-free graph. The *r-closure* of G, denoted by $cl^r(G)$, is the graph defined by a sequence of graphs $G_1, G_2, \ldots, G_t$, and vertices $x_1, x_2, \ldots, x_{t-1}$ such that

(1) $G_1 = G$, $G_t = cl^r(G)$;

(2) x_i is an r-eligible vertex of G_i, $G_{i+1} = (G_i)'_{x_i}$, $1 \leq i \leq t-1$; and

(3) $cl^r(G)$ has no r-eligible vertices.

A claw-free graph is said to be *r-closed* if it has no r-eligible vertices.

Theorem 1.18 (Ryjáček [32]). *Let G be a claw-free graph. Then*

(1) *the r-closure $cl^r(G)$ is well-defined;*

(2) *there is a triangle-free graph H such that $cl^r(G)$ is the line graph of H; and*

(3) $c(G) = c(cl^r(G))$.

Let $\mathcal{P}$ be a property of graphs. $\mathcal{P}$ is said to be *stable under the r-closure* (or simply, *r-stable*), if for every claw-free graph with property $\mathcal{P}$, its r-closure also satisfies the property $\mathcal{P}$. It is easy to deduce from the above results that the properties hamiltonicity and non-hamiltonicity are r-stable.

On the r-stability of the property S-freeness for some graph S, Brousek et al. proved the following result. Here H denotes the graph obtained from two triangles by identifying two vertices, one of each of the triangles (see Figure 1.4; this graph is sometimes called an *hourglass*).

Theorem 1.19 (Brousek, Ryjáček and Schiermeyer [16]). *Let S be an r-closed connected claw-free graph. Then the class of $\{K_{1,3}, S\}$-free graphs is r-stable if and only if*

$$S = \{C_3, H\} \cup \{P_i : i \geq 3\} \cup \{Z_i : i \geq 1\} \cup \{N_{i,j,k} : i, j, k \geq 1\}.$$

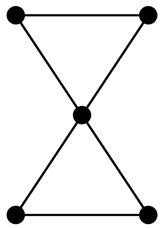


Figure 1.4: Graph H

By the above closure theory, when one considers the hamiltonicity of $\{K_{1,3}, S\}$-free graphs, for some graph S in the above theorem, it is convenient to consider the r-closure of the graphs. Using this closure concept, several researchers obtained many results (see, e.g., [10, 15, 16]).

As shown in [13], the properties of being homogeneously traceable or pancyclic are not r-stable in general. Thus the above closure theory cannot be applied in a straightforward way when we consider these properties of graphs.

In order to study the hamiltonicity of claw-heavy graphs, Čada proposed an alternative for the above closure theory.

Let G be a claw-heavy graph on n vertices and let $x \in V(G)$. Let G' be the graph obtained from G by adding the missing edges uv with $u, v \in N(x)$ and $d(u) + d(v) \geq n$. We call x a *c-eligible* vertex of G if $N(x)$ is not a clique and one of the following is true:

(1) $N_{G'}(x)$ induces a connected graph in G'; or

(2) $N_{G'}(x)$ consists of two cliques C_1 and C_2, and there is a vertex z nonadjacent to x such that $d(x) + d(z) \geq n$ and $zy_1, zy_2 \in E(G)$ for some $y_1 \in V(C_1)$ and $y_2 \in V(C_2)$.

Theorem 1.20 (Čada [17]). *Let G be a claw-heavy graph, and let x be a c-eligible vertex of G. Then*

(1) *the graph G'_x is claw-heavy; and*

(2) $c(G'_x) = c(G)$.

Similarly to the r-closure, we can define corresponding concepts as follows.

Let G be a claw-heavy graph. The *c-closure* of G, denoted by $cl^c(G)$, is the graph defined by a sequence of graphs $G_1, G_2, \ldots, G_t$, and vertices $x_1, x_2, \ldots, x_{t-1}$ such that
(1) $G_1 = G$, $G_t = cl^c(G)$;
(2) x_i is a c-eligible vertex of G_i, $G_{i+1} = (G_i)'_{x_i}$, $1 \leq i \leq t-1$; and
(3) $cl^c(G)$ has no c-eligible vertices.

A claw-heavy graph is said to be *c-closed* if it has no c-eligible vertices.

Theorem 1.21 (Čada [17]). *Let G be a claw-heavy graph. Then*
(1) *the c-closure $cl^c(G)$ is well-defined;*
(2) *there is a triangle-free graph H such that $cl^c(G)$ is the line graph of H; and*
(3) $c(G) = c(cl^c(G))$.

Let $\mathcal{P}$ be a property of graphs. $\mathcal{P}$ is said to be *stable under the c-closure* (or simply, *c-stable*), if for every claw-free graph with property $\mathcal{P}$, its c-closure also satisfies the property $\mathcal{P}$.

In contrast to the results on the r-closure, apart from several trivial graphs, the property being S-heavy is generally not c-stable for any c-closed connected claw-free graph S. For this reason, when we consider heavy subgraph pairs for hamiltonian properties, the alternative closure theory is also difficult to apply. This is the reason why most of the proofs in this thesis require new methods for obtaining the hamiltonian properties of claw-heavy graphs. We found several fit-for-purpose methods for the case when no closure theory seemed to be applicable. We refer to Chapters 5, 6 and 7 for more details.

1.4 Other related properties

There are many other graph properties that are interesting to consider in the context of forbidden and heavy subgraph conditions. Some of them have been researched with respect to forbidden subgraph conditions, but for heavy subgraph conditions we do not know of any complete characterizations of pairs of heavy subgraphs for other graph properties. Below we mention some graph properties that could be interesting for future research.

A. On the existence of dominating cycles

Let G be a graph. A *dominating cycle* of G is a cycle C such that every component of $G - C$ is an isolated vertex. Note that if a graph is hamiltonian, then it obviously has a dominating cycle.

Problem 1.1. Which pairs of connected graphs $\{R, S\}$ imply that every 2-connected $\{R, S\}$-free (or $\{R, S\}$-heavy) graph has a dominating cycle?

B. On the existence of 2-factors

A *2-factor* of a graph G is the union of some pairwise vertex-disjoint cycles such that every vertex of G is contained in one of the cycles. With respect to forbidden subgraph pairs for the existence of a 2-factor in a 2-connected graph (on at least 10 vertices), a complete characterization has been given in [23].

Theorem 1.22 (Faudree et al. [23]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph on at least 10 vertices. Then G being $\{R, S\}$-free implies G has a 2-factor if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $B_{1,4}$ or $N_{1,1,3}$, or $R = K_{1,4}$ and $S = P_4$.*

The following problem is open and seems to be suitable for future research.

Problem 1.2. Which pairs of connected graphs $\{R, S\}$ imply that every 2-connected $\{R, S\}$-heavy graph has a 2-factor?

Note that 2-connectivity, even connectivity, is not a necessary condition for the existence of a 2-factor, so one might consider relaxing this condition. The existence of a 2-factor in a given graph can be decided in polynomial time. In this respect, this graph property looks less interesting than the other properties, but it would still be interesting to know how much the heavy subgraph pairs for this property differ from the forbidden subgraph pairs.

C. On pancyclicity of 3-connected graphs

With respect to forbidden pairs of graphs that imply a 3-connected graph is pancyclic, Gould et al. gave a complete characterization. In the following theorem, L is the graph obtained by joining two vertices of two disjoint triangles by an edge (see Figure 1.5).

Theorem 1.23 (Gould et al. [29]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 3-connected graph. Then G being $\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of L, P_7, Z_4, $B_{1,3}$, $B_{2,2}$ or $N_{1,1,2}$.*

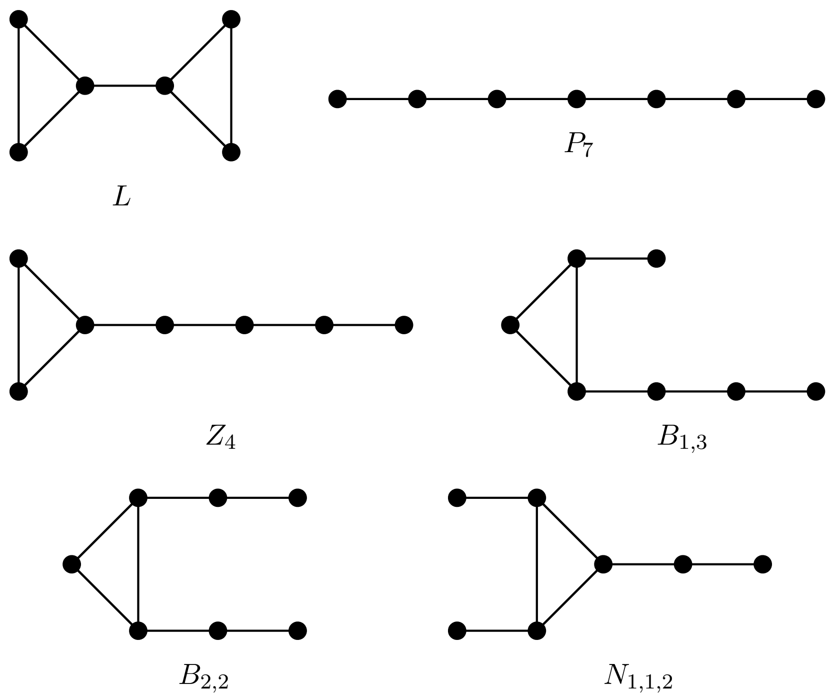


Figure 1.5: Graphs L, P_7, Z_4, $B_{1,3}$, $B_{2,2}$ and $N_{1,1,2}$

Note that 3-connectivity is not a necessary condition for pancyclicity, so imposing 3-connectivity seems a bit artificial. Nevertheless, by doing so the forbidden subgraphs become larger, so the result applies to a richer class of (3-connected) graphs. We do not know any counterpart of this result for o_1-heavy pairs of subgraphs.

Problem 1.3. Which pairs of connected graphs $\{R, S\}$ imply that every 3-connected $\{R, S\}$-o_1-heavy graph is pancyclic?

C. On Hamilton-connectedness of 3-connected graphs

If a graph G has a vertex cut with two vertices, then G cannot have a

Hamilton path connecting these two vertices. This implies that every Hamilton-connected graph (on at least 4 vertices) is 3-connected, i.e., the necessary connectivity condition for Hamilton-connectedness is 3-connectivity. Several results are known involving forbidden pairs of graphs that imply a 3-connected graph is Hamilton-connected (see, e.g., [9, 24]). These papers also contain graph families of non-Hamilton-connected graphs that restrict the pairs considerably, but as far as we know there is no complete characterization of the forbidden pairs for this problem.

Note that the complete balanced bipartite graph $K_{n/2,n/2}$ is not Hamilton-connected, and every two nonadjacent vertices of it have degree sum n. This implies that the necessary degree sum for Hamilton-connectedness is $n + 1$. To finish the introduction, we propose the following problem.

Problem 1.4. Which pairs of connected graphs $\{R, S\}$ imply that every 3-connected $\{R, S\}$-free (or $\{R, S\}$-o_1-heavy) graph is Hamilton-connected?

Chapter 2

Heavy subgraphs for heavy longest cycles

2.1 Introduction

Let G be a graph on n vertices. A vertex v is called a *heavy vertex* of G if $d(v) \geq n/2$, and a cycle C is called a *heavy* cycle of G if C contains all heavy vertices of G.

The following theorem on the existence of heavy cycles in graphs is well-known.

Theorem 2.1 (Bollobás and Brightwell [6], Shi [33])**.** *Every 2-connected graph has a heavy cycle.*

In this chapter, we first characterize the separable graphs that contain no heavy cycles.

Let $G = (V, E)$ be a graph, $v \in V$, and $e \in E$. We use $G - v$ to denote the graph obtained from G by deleting v and all the edges incident with v, and $G - e$ to denote the graph obtained from G by deleting e.

We first obtain a structural result on the distribution of heavy vertices in a connected graph that does not contain a heavy cycle.

Theorem 2.2. *Let G be a connected graph on n vertices and suppose that G contains no heavy cycle. Then G has at most two heavy vertices. Moreover,*

(1) *if G contains no heavy vertices, then G is a tree;*

(2) *if G contains precisely one heavy vertex, say x, then $G - x$ contains at least $n/2$ components, and each component of $G - x$ contains exactly one neighbor of x; and*

(3) *if G has exactly two heavy vertices, say x and y, then $xy \in E(G)$ and xy is a cut edge of G, n is even and both components of $G - xy$ have $n/2$ vertices, and x (or y, respectively) is adjacent to every other vertex of the component containing x (or y, respectively).*

Briefly stated, (3) of the above theorem means that G is a spanning supergraph of T_1 and a spanning subgraph of T_2, with T_1 and T_2 as indicated in Figure 2.1.

We postpone the proof of Theorem 2.2 to Section 2.3.

In general, a longest cycle of a graph may not be a heavy cycle (see, e.g., Figure 2.2). In this chapter, we mainly consider heavy subgraph conditions for longest cycles to be heavy. First, consider the following theorem of Fan [22].

Theorem 2.3 (Fan [22]). *Let G be a 2-connected graph. If $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices u, v with distance 2 in G, then G is hamiltonian.*

This theorem implies that every 2-connected P_3-heavy graph has a Hamilton cycle, which is of course a heavy cycle. In fact, we will prove the following theorem in Section 2.4.

Theorem 2.4. *If G is a 2-connected $K_{1,4}$-heavy graph, and C is a longest cycle of G, then C is a heavy cycle of G.*

Note that $K_{1,3}$ is an induced subgraph of $K_{1,4}$. So any longest cycle of a 2-connected $K_{1,3}$-heavy graph is heavy. In fact, P_3, $K_{1,3}$ and $K_{1,4}$ are the only connected graphs satisfying this property, as shown by the next result.

Theorem 2.5. *Let S be a connected graph on at least 3 vertices and let G be a 2-connected graph. Then G being S-free (or S-heavy) implies every longest cycle of G is a heavy cycle, if and only if $S = P_3$, $K_{1,3}$ or $K_{1,4}$.*

The 'if' part of the proof of this theorem follows from Theorem 2.4 immediately. We will prove the 'only-if' part in Section 2.5.

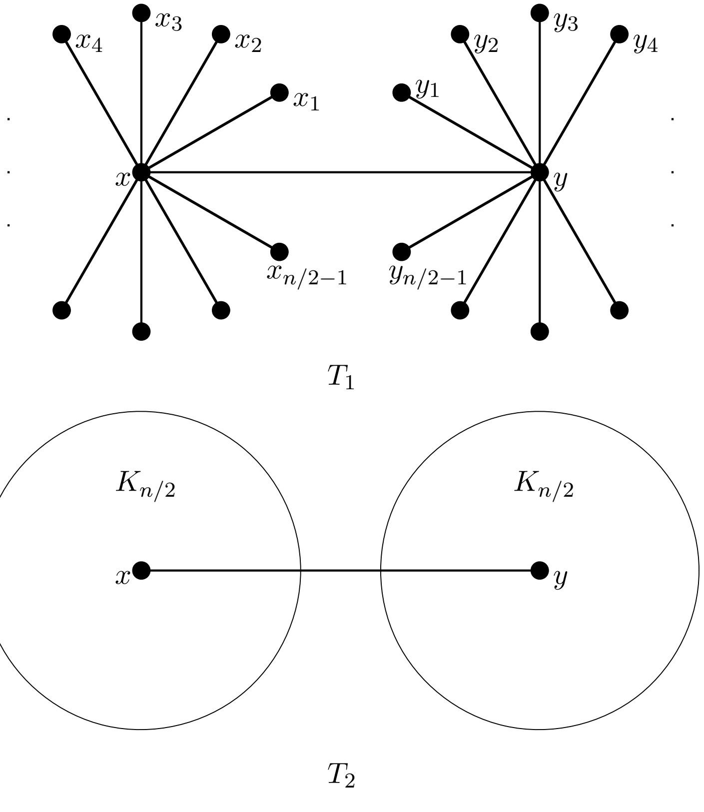


Figure 2.1: Extremal graphs with two heavy vertices and no heavy cycles

2.2 Some preliminaries

We first give some additional terminology and notation.

Let s and t be two integers with $s \leq t$, and let x_i, $s \leq i \leq t$, be vertices of a graph. We use $[x_s, x_t]$ to denote the set of vertices $\{x_i : s \leq i \leq t\}$.

Let P be a path and $x, y \in V(P)$. We use $P[x, y]$ to denote the subpath of P from x to y. Let C be a cycle with a given orientation and $x, y \in V(C)$. We use $\overrightarrow{C}[x, y]$ and $\overleftarrow{C}[y, x]$ to denote the (x, y)-path on C traversed in the same or opposite direction with respect to the given orientation of C, respectively.

Let G be a graph on n vertices and let $k \geq 3$ be an integer. We call a circular sequence of vertices $C = v_1v_2 \cdots v_kv_1$ an *Ore-cycle* (or briefly, an *o-cycle*) of G, if for all i with $1 \leq i \leq k$, either $v_iv_{i+1} \in E(G)$ or $d(v_i)+d(v_{i+1}) \geq n$, where $v_{k+1} = v_1$. The *deficit* of C is defined as $\mathrm{def}(C) = |\{i : v_iv_{i+1} \notin E(G)$ with $1 \leq i \leq k\}|$. Thus a cycle is an o-cycle with deficit degree 0.

Similarly, we can define *o-paths* of G.

Now, we prove the following lemma on o-cycles.

Lemma 1. *Let G be a graph and let C be an o-cycle of G. Then there exists a cycle of G containing all the vertices of $V(C)$.*

Proof. Assume the opposite. Let C' be an o-cycle containing all the vertices of $V(C)$ such that $\mathrm{def}(C')$ is as small as possible. Then $\mathrm{def}(C') \geq 1$. Without loss of generality, we suppose that $C' = v_1v_2 \cdots v_kv_1$, where $v_1v_k \notin E(G)$ and $d(v_1) + d(v_k) \geq n$. We use P to denote the o-path $P = v_1v_2 \cdots v_k$.

If v_1 and v_k have a common neighbor in $V(G)\backslash V(P)$, denote it by x. Then $C'' = Pv_kxv_1$ is an o-cycle containing all the vertices of $V(C)$, but with deficit degree smaller than $\mathrm{def}(C')$, a contradiction.

So we assume that $N_{G-P}(v_1) \cap N_{G-P}(v_k) = \emptyset$. Then $d_P(v_1) + d_P(v_k) \geq |V(P)|$, since $d(v_1) + d(v_k) \geq n$. Thus, there exists an integer i with $2 \leq i \leq k-1$ such that $v_i \in N_P(v_1)$ and $v_{i-1} \in N_P(v_k)$. But then $C'' = P[v_1, v_{i-1}]v_{i-1}v_kP[v_k, v_i]v_iv_1$ is an o-cycle containing all the vertices of $V(C)$, and with deficit degree smaller than $\mathrm{def}(C')$, a contradiction. □

Note that Theorem 2.1 can easily be deduced from Lemma 1.

Let P be an (x, y)-path (or o-path) of G. If the number of vertices of P is more than that of a longest cycle of G, then, by Lemma 1, we have $xy \notin E(G)$ and $d(x) + d(y) < n$.

In the following, we use $\widetilde{E}(G)$ to denote the set $\{uv : uv \in E(G)$ or $d(u) + d(v) \geq n\}$.

2.3 Proof of Theorem 2.2

We assume that G is a connected graph on n vertices, and we suppose that G contains no heavy cycle. If G contains at least three heavy vertices, then let $X = \{x_1, x_2, \ldots, x_k\}$ be the set of heavy vertices of G, where $k \geq 3$. Then $C = x_1 x_2 \cdots x_k x_1$ is an o-cycle. By Lemma 1, there exists a cycle containing all the vertices of X, which is a heavy cycle, a contradiction. Thus G contains at most two heavy vertices.

Suppose that G contains no heavy vertices, but that G has a cycle C. Then C is a heavy cycle of G, a contradiction. So if G contains no heavy vertices, then G is a tree, proving (i) of Theorem 2.2. Next we consider the two remaining cases: G contains exactly one or exactly two heavy vertices.

Case 1. G contains exactly one heavy vertex.

Let x be the heavy vertex of G, and let H be a component of $G - x$. Since G is connected, $N_H(x) \neq \emptyset$. If $|N_H(x)| \geq 2$, then let x_1 and x_2 be two vertices in $N_H(x)$, and let P be an (x_1, x_2)-path in H. Then $C = P x_2 x x_1$ is a cycle containing x, which is a heavy cycle, a contradiction. Thus $|N_H(x)| = 1$.

Since $d(x) \geq n/2$, we conclude that $G - x$ contains at least $n/2$ components, proving (ii) of Theorem 2.2.

Case 2. G contains exactly two heavy vertices.

Let x and y be the two heavy vertices, and let P be a longest (x, y)-path of G. If $|V(P)| \geq 3$, then $C' = xPyx$ is an o-cycle of G. By Lemma 1, there exists a cycle containing all the vertices of $V(C')$, which is a heavy cycle, a contradiction. Thus $|V(P)| = 2$, implying that $xy \in E(G)$ and that xy is a cut edge of G.

Let H_x and H_y be the components of $G - xy$ containing x and y, respectively. Since $d(x) \geq n/2$ and $xy' \notin E(G)$ for all $y' \in V(H_y) \backslash \{y\}$, we get that

$|V(H_y)| \le n/2$, and similarly, $|V(H_x)| \le n/2$. This implies that n is even and $|V(H_x)| = |V(H_y)| = n/2$.

Since $d(x) \ge n/2$ and $|V(H_x)| = n/2$, we get that $xx' \in E(G)$ for every $x' \in V(H_x)\backslash\{x\}$. Similarly, $yy' \in E(G)$ for every $y' \in V(H_y)\backslash\{y\}$.

This completes the proof of Theorem 2.2.

2.4 Proof of Theorem 2.4

We assume that G is a 2-connected $K_{1,4}$-heavy graph on n vertices, that C is a longest cycle of G, and that c is the length of C. We give an orientation to C. We are going to prove that C is a heavy cycle of G. Let x be a vertex in $V(G) \setminus V(C)$. It is sufficient to prove that $d(x) < n/2$.

Let H be the component of $G - V(C)$ containing x. Then all the neighbors of x are in $V(C) \cup V(H)$. Let $h = |V(H)|$. Noting that x is not a neighbor of itself, we have $d_H(x) < h$. We are going to prove a number of useful claims, the first of which is easy to check.

Claim 1. If v_1, v_2 are two vertices of $V(C)$ with $v_1v_2 \in E(C)$, then either $xv_1 \notin E(G)$ or $xv_2 \notin E(G)$.

Proof. Otherwise, $C - v_1v_2 \cup v_1xv_2$ (with the obvious meaning) is a longer cycle than C, a contradiction. □

By Claim 1, if P is a subpath of C, then $d_P(x) \le \lceil |V(P)|/2 \rceil$.

By the 2-connectedness of G, there exists a (u_0, v_0)-path (and thus, a (u_0, v_0)-o-path) passing through x which is internally-disjoint with C, where $u_0, v_0 \in V(C)$. We choose such an o-path $Q = x_{-k}x_{-k+1}\cdots x_{-1}xx_1\cdots x_\ell$ such that

(1) $x_{\pm 1} \in N(x)$; and

(2) $|V(Q) \cap N_H(x)|$ is as large as possible,

where $x_{-k} \in V(C)$ and $x_\ell \in V(C)$.

Claim 2. Q contains at least half of the vertices in $N_H(x)$.

Proof. If $d_H(x) = 0$, the assertion is obvious. So we assume that $d_H(x) \ge 1$.

Suppose that $|N_H(x) \cap V(Q)| < d_H(x)/2$. Then $|N_H(x) \backslash V(Q)| \geq \lceil d_H(x)/2 \rceil \geq 1$. We first prove four subclaims.

Claim 2.1. For every $x' \in N_H(x) \backslash V(Q)$, $x'x_1 \notin \widetilde{E}(G)$ and $x'x_{-1} \notin \widetilde{E}(G)$.

Proof. If $x'x_1 \in \widetilde{E}(G)$, then $Q' = Q[x_{-k}, x]xx'x_1Q[x_1, x_l]$ is an o-path containing more vertices of $N_H(x)$ than Q, a contradiction. Thus $x'x_1 \notin \widetilde{E}(G)$.

The second assertion can be proved similarly. □

Claim 2.2. $x_{-1}x_1 \in \widetilde{E}(G)$.

Proof. Suppose that $x_{-1}x_1 \notin \widetilde{E}(G)$. Let x'_i, x'_j be any pair of vertices in $N_H(x) \backslash V(Q)$. By Claim 2.1, $x'_ix_{\pm 1} \notin \widetilde{E}(G)$ and $x'_jx_{\pm 1} \notin \widetilde{E}(G)$. Since G is a $K_{1,4}$-heavy graph, $x'_ix'_j \in \widetilde{E}(G)$.

By the 2-connectedness of G, there is a path from $N_H(x) \backslash V(Q)$ to $V(C) \cup V(Q)$ not passing through x. Let $R' = y_1y_2 \cdots y_r$ be such a path, where $y_1 \in N_H(x) \backslash V(Q)$ and $y_r \in V(C) \cup V(Q) \backslash \{x\}$. Let R be an o-path from x to y_1 passing through all the vertices in $N_H(x) \backslash V(Q)$.

If $y_r \in V(C) \backslash \{x_{-k}, x_l\}$, then $Q' = Q[x_{-k}, x]xRy_1R'$ is an o-path containing at least half of the vertices of $N_H(x)$, a contradiction.

If $y_r \in V(Q[x_1, x_l])$, then $Q' = Q[x_{-k}, x]xRy_1R'y_rQ[y_r, x_l]$ is an o-path containing at least half of the vertices of $N_H(x)$, a contradiction.

If $y_r \in V(Q[x_{-k}, x_{-1}])$, then we can prove the result analogously.

Thus the claim holds. □

Now, we choose an o-path $R = xx'_1x'_2 \cdots x'_r$ which is internally-disjoint with $C \cup Q$, where $x'_r \in V(C) \cup V(Q) \backslash \{x\}$ such that

(1) $x'_1 \in N(x)$; and
(2) $|V(R) \cap (N_H(x) \backslash V(Q))|$ is as large as possible.

Claim 2.3. R contains at least half of the vertices of $N_H(x) \backslash V(Q)$.

Proof. Note that $d_{H-Q}(x) \geq 1$. It is easy to check that $x'_1 \in N_H(x) \backslash V(Q)$. By Claim 2.1, $x'_1x_1 \notin \widetilde{E}(G)$.

Suppose that $|V(R) \cap (N_H(x) \backslash V(Q))| < d_{H-Q}(x)/2$. Let $N_H(x) \backslash V(Q) \backslash V(R) = \{x''_1, x''_2, \ldots, x''_s\}$, where $s \geq \lceil d_{H-Q}(x)/2 \rceil$.

For every vertex $x''_i \in N_H(x) \backslash V(Q) \backslash V(R)$, by Claim 2.1, $x''_ix_1 \notin \widetilde{E}(G)$. Similarly, we can prove that $x''_ix'_1 \notin \widetilde{E}(G)$.

For any pair of vertices $x''_i, x''_j \in N_H(x)\backslash V(Q)\backslash V(R)$, $x''_i x_1 \notin \widetilde{E}(G)$, $x''_i x'_1 \notin \widetilde{E}(G)$, $x''_j x_1 \notin \widetilde{E}(G)$, $x''_j x'_1 \notin \widetilde{E}(G)$ and $x'_1 x_1 \notin \widetilde{E}(G)$. Since G is $K_{1,4}$-heavy, we conclude that $x''_i x''_j \in \widetilde{E}(G)$.

By the 2-connectedness of G, there is a path from $N_H(x)\backslash V(Q)\backslash V(R)$ to $V(C) \cup V(Q)$ not passing through x. Let $T' = y_1 y_2 \dots y_t$ be such a path, where $y_1 \in N_H(x)\backslash V(Q)\backslash V(R)$ and $y_t \in V(C) \cup V(Q)\backslash\{x\}$. Let T be an o-path from x to y_1 passing through all the vertices in $N_H(x)\backslash V(Q)\backslash V(R)$. Then $R' = Ty_1T'$ is an o-path from x to $V(C) \cup V(Q)\backslash\{x\}$ containing at least half of the vertices of $N_H(x)\backslash V(Q)$, a contradiction. □

By Claim 2.3, R contains at least one quarter of the vertices of $N_H(x)$.

Claim 2.4. $x'_r \in V(C)\backslash\{x_{-k}, x_\ell\}$.

Proof. Assume the opposite. Without loss of generality, we assume that $x'_r \in [x_1, x_\ell]$.

If $x'_r = x_1$, then $Q' = Q[x_{-k}, x]xRx_1Q[x_1, x_l]$ is an o-path containing more vertices of $N_H(x)$ than Q, a contradiction.

If $x'_r = x_i$, where $2 \le i \le l$, then let x_j be the last vertex in $[x_1, x_{i-1}]$ such that $x_j \in N(x)$. Then $Q' = Q[x_{-k}, x_{-1}]x_{-1}x_1Q[x_1, x_j]x_jxRx'_rQ[x'_r, x_\ell]$ is an o-path containing more vertices of $N_H(x)$ than Q, a contradiction.

Thus $x'_r \in V(C)\backslash\{x_{-k}, x_\ell\}$. □

If $Q[x, x_\ell]$ contains less than one quarter of the vertices in $N_H(x)$, then $Q' = Q[x_{-k}, x]xR$ is an o-path containing more vertices of $N_H(x)$ than Q, a contradiction. This implies that $Q[x, x_\ell]$ contains at least one quarter of the vertices of $N_H(x)$. Similarly, $Q[x_{-k}, x]$ contains at least one quarter of the vertices of $N_H(x)$. Thus Q contains at least half of the vertices of $N_H(x)$, a contradiction. This completes the proof of Claim 2. □

By Claim 2, $k + \ell - 2 \ge d_H(x)/2$.

Let $u_0 = x_{-k} \in V(C)$ and $v_0 = x_\ell \in V(C)$. We assume that the length of $\overrightarrow{C}[v_0, u_0]$ is r_1+1, and that the length of $\overrightarrow{C}[u_0, v_0]$ is r_2+1, where $r_1+r_2+2 = c$. We use $\overrightarrow{C} = v_0v_1v_2 \cdots v_{r_1}u_0v_{-r_2}v_{-r_2+1} \cdots v_{-1}v_0$ to denote C with the given orientation, and $\overleftarrow{C} = u_0u_1u_2 \cdots u_{r_1}v_0u_{-r_2}u_{-r_2+1} \cdots u_{-1}u_0$ to denote C with the opposite orientation, where $v_i = u_{r_1+1-i}$ and $v_{-j} = u_{-r_2-1+j}$.

Claim 3. $r_1 \geq k+\ell-1$, and for every vertex $v_s \in [v_1, v_\ell]$, $xv_s \notin E(G)$, and for every vertex $u_t \in [u_1, u_k]$, $xu_t \notin E(G)$.

Proof. Note that Q contains $k+\ell-1$ vertices of $V(H)$. If $r_1 < k+\ell-1$, then $C' = Qv_0\overleftarrow{C}[v_0, u_0]u_0$ is a longer o-cycle than C. By Lemma 1, there exists a cycle containing all the vertices of $V(C')$, a contradiction. Thus, $r_1 \geq k+\ell-1$.

If $xv_s \in E(G)$, where $v_s \in [v_1, v_\ell]$, then $C' = \overrightarrow{C}[v_s, v_0]v_0Q[v_0, x]xv_s$ is an o-cycle containing all the vertices of $V(C)\backslash[v_1, v_{s-1}] \cup V(Q[x, x_{l-1}])$, and $|V(C')| > c$, a contradiction.

If $xu_t \in E(G)$, where $u_t \in [u_1, u_k]$, then we can prove the result analogously. This completes the proof of Claim 3. □

Similarly, we can prove the following claim.

Claim 4. $r_2 \geq k+\ell-1$, and for every vertex $v_{-s} \in [v_{-\ell}, v_{-1}]$, $xv_{-s} \notin E(G)$, and for every vertex $u_{-t} \in [u_{-k}, u_{-1}]$, $xu_{-t} \notin E(G)$.

Let $d_1 = d_{\overrightarrow{C}[v_1,u_1]}(x)$ and $d_2 = d_{\overleftarrow{C}[v_{-1},u_{-1}]}(x)$. Then $d_C(x) \leq d_1 + d_2 + 2$.

Claim 5. $d_1 \leq (r_1 - (k+\ell) + 1)/2$ and $d_2 \leq (r_2 - (k+\ell) + 1)/2$.

Proof. If $r_1 = k+\ell-1$, then by Claim 3, $d_1 = 0$. So we assume that $r_1 \geq k+\ell$.

By Claim 3, $d_1 = d_{\overrightarrow{C}[v_{\ell+1},u_{k+1}]}(x)$. By Claim 1, $d_1 \leq \lceil (r_1 - (k+\ell))/2 \rceil \leq (r_1 - (k+\ell) + 1)/2$.

The second assertion can be proved analogously. □

By Claim 5,

$$d_C(x) \leq d_1 + d_2 + 2 \leq (r_1 + r_2 + 2 - 2(k+\ell))/2 + 2 = c/2 - (k+\ell-2).$$

Noting that $k+\ell-2 \geq d_H(x)/2$, we get $d_C(x) \leq (c - d_H(x))/2$. Thus $d(x) = d_C(x) + d_H(x) \leq (c + d_H(x))/2 < (c+h)/2 \leq n/2$.

This completes the proof of Theorem 2.4.

2.5 The 'only-if' part of the proof of Theorem 2.5

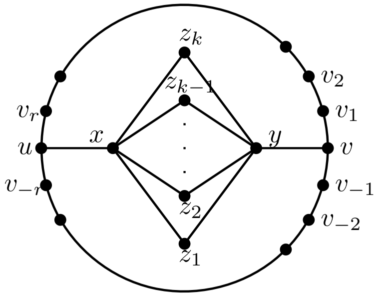
Noting that an S-free graph is also S-heavy, it suffices to prove that a longest cycle of a 2-connected S-free graph is not necessarily a heavy cycle if $S \neq P_3$, $K_{1,3}$ and $K_{1,4}$.

First consider the following fact: if a connected graph S on at least 3 vertices is not P_3, $K_{1,3}$ or $K_{1,4}$, then S must contain K_3, P_4, C_4 or $K_{1,5}$ as an induced subgraph. Thus we only need to show that not every longest cycle in a K_3-free, P_4-free, C_4-free or $K_{1,5}$-free graph is heavy.

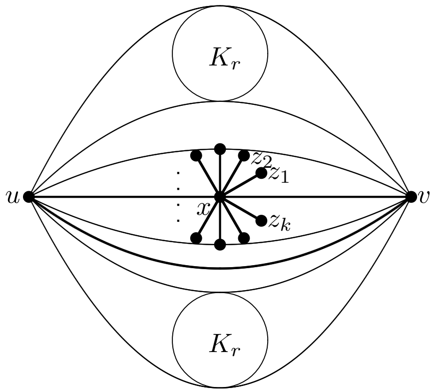
We construct three (classes of) graphs as sketched and indicated by G_1, G_2 and G_3 in Figure 2.2.

The structure of a graph G_1 of type 1 is clear from the figure: all edges are drawn in the figure, except for the dots in the middle that indicate the missing vertices z_i and edges xz_i and yz_i, and the longer circular segments indicating connecting path along the outer cycle. With the right choice of the parameter values k and r, the outer cycle is the longest cycle and this is clearly not a heavy cycle because it misses the heavy vertices x and y. In a graph G_2 of type 2, the subgraph $G_2[\{x\} \cup [z_1, z_k]]$ is a star $K_{1,k}$, and u and v are adjacent to all the vertices of the $K_{1,k}$ and of the two K_r's (note that also $uv \in E(G_2)$). With the right choice of the parameter values k and r, any longest cycle passes through u and v, picking up all the vertices of the two K_r's, but missing the heavy vertex x. In a graph G_3 of type 3, x and y are adjacent to all the vertices of the three K_k's. With the right choice of the parameter values k and r, the outer cycle is the longest cycle and it clearly misses the heavy vertices x and y.

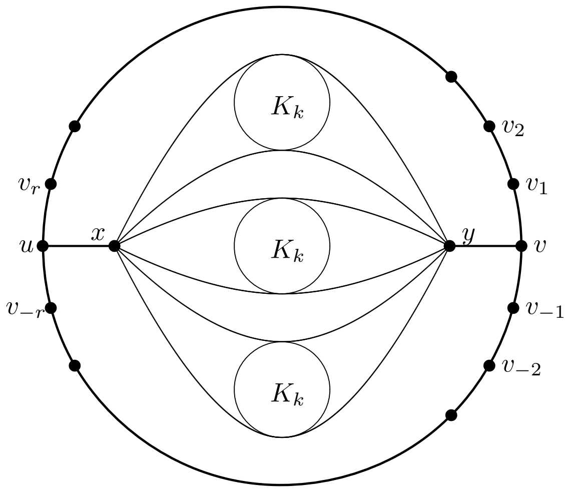
Note that G_1 is K_3-free, G_2 is P_4-free and C_4-free and G_3 is $K_{1,5}$-free. This completes the proof of the 'only-if' part of Theorem 2.5.



G_1 ($r \geq 4$ and $k \geq 2r + 2$)



G_2 ($r \geq 4$ and $k \geq 2r - 1$)



G_3 ($r \geq 11$ and $(2r + 2)/3 \leq k \leq r - 3$)

Figure 2.2: Graphs G_1, G_2 and G_3

Chapter 3

Heavy pairs for traceability

3.1 Introduction

A graph is *traceable* if it contains a *Hamilton path*, i.e., a path containing all its vertices. In this chapter, we consider heavy subgraph conditions for the traceability of connected graphs.

If a graph G is connected and P_3-free, then it is a complete graph and it is therefore trivially traceable. In fact, P_3 is the only single subgraph with this property. The following theorem on forbidden pairs of subgraphs for traceability is well-known.

Theorem 3.1 (Duffus, Jacobson and Gould [21]). *If G is a connected $\{K_{1,3}, N\}$-free graph, then G is traceable.*

Obviously, if H is an induced subgraph of N, then an $\{K_{1,3}, H\}$-free connected graph is also traceable. Faudree and Gould proved that these are the only forbidden pairs with such property. We refer to Figure 3.1 for an illustration of the graphs appearing in the next result.

Theorem 3.2 (Faudree and Gould [24]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$, P_4, Z_1, B or N.*

A natural question is to consider o_{-1}-heavy subgraph conditions for a graph

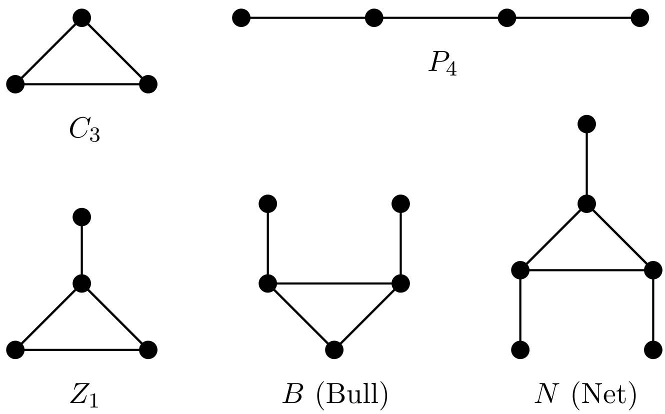


Figure 3.1: Graphs C_3, P_4, Z_1, B and N

to be traceable. First, we will prove in Section 3.4 that every 2-connected P_3-o_{-1}-heavy graph is traceable.

Theorem 3.3. *If G is a connected P_3-o_{-1}-heavy graph, then G is traceable.*

It is not difficult to see that P_3 is the only connected graph S such that every connected S-o_{-1}-heavy graph is traceable. Now we consider which two connected graphs R and S other than P_3 imply that every connected $\{R, S\}$-o_{-1}-heavy graph is traceable. In fact, perhaps surprisingly, as we will show below, there is only one such pair of subgraphs.

Theorem 3.4. *Let R and S be connected graphs with $R, S \neq P_3$, and let G be a connected graph. Then G being $\{R, S\}$-o_{-1}-heavy implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$.*

Since C_3 is a clique, saying that a graph is C_3-o_{-1}-heavy is equivalent to saying that it is C_3-free. Thus for the 'if' part of Theorem 3.4, we only need to prove that every connected claw-o_{-1}-heavy and C_3-free graph is traceable. In fact, we can prove the following stronger theorem.

Theorem 3.5. *If G is a connected claw-o_{-1}-heavy and Z_1-free graph, then G is traceable.*

We postpone the proof of Theorem 3.5 to Section 3.5. In Section 3.6 we prove the following theorem, which gives another forbidden subgraph for a connected claw-o_{-1}-heavy graph to be traceable.

Theorem 3.6. *If G is a connected claw-o_{-1}-heavy and P_4-free graph, then G is traceable.*

In fact, these are the only forbidden subgraphs satisfying such property.

Theorem 3.7. *Let S be connected graphs with $S \neq P_3$, and let G be a connected claw-o_{-1}-heavy graph. Then G being S-free implies G is traceable if and only if $S = C_3$, Z_1 or P_4.*

We prove the 'only-if' part of Theorems 3.4 and 3.7 in Section 3.2.

3.2 The 'only-if' part of Theorems 3.4 and 3.7

We construct two families of non-traceable graphs as depicted in Figure 3.2. Since all the graphs have at least three vertices with degree 1, it is obvious that none of these graphs are traceable.

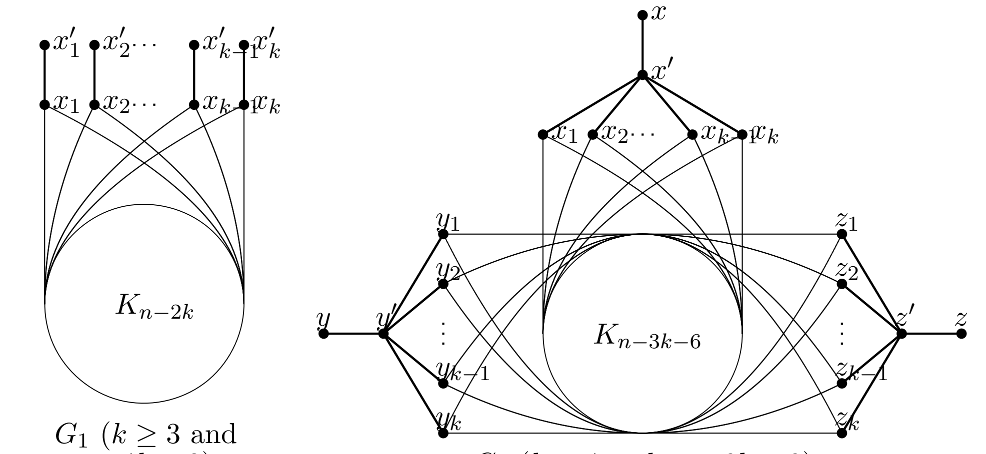
Let R and S be two connected graphs other than P_3 such that every connected $\{R,S\}$-o_{-1}-heavy graph is traceable. Then by Theorem 3.2, up to symmetry, $R = K_{1,3}$ and S is C_3, P_4, Z_1, B or N. Note that G_1 is $\{K_{1,3},P_4\}$-o_{-1}-heavy and that G_2 is $\{K_{1,3},Z_1\}$-o_{-1}-heavy. Hence S must be C_3. This completes the proof of the 'only-if' part of Theorem 3.4.

Let S be a connected graph other than P_3 such that every connected claw-o_{-1}-heavy and S-free graph is traceable. By Theorem 3.2, S must be C_3, P_4, Z_1, B or N. Note that G_1 is B-free. Hence S must be C_3, P_4 or Z_1. This completes the proof of the 'only-if' part of Theorem 3.7.

3.3 Some preliminaries

We first give some additional terminology and notation.

Let G be a graph, let P be a path of G, and let $x, y \in V(P)$. We use $P[x,y]$ to denote the subpath of P from x to y.



G_1 ($k \geq 3$ and $n \geq 4k-3$) G_2 ($k \geq 4$ and $n \geq 6k+9$)

Figure 3.2: Two families of non-traceable graphs

Let G be a graph on n vertices and let k be an integer. We call a sequence of vertices $P = v_1v_2\cdots v_k$ an *o_{-1}-path* of G, if for all $i \in [1, k-1]$, either $v_iv_{i+1} \in E(G)$ or $d(v_i) + d(v_{i+1}) \geq n-1$. The *deficit* of P is defined by $\mathrm{def}(P) = |\{i \in [1, k-1] : v_iv_{i+1} \notin E(G)\}|$. Thus a path is an o_{-1}-path with deficit 0.

We first prove the following lemma on o_{-1}-paths.

Lemma 1. *Let G be a graph and let P be an o_{-1}-path of G. Then there exists a path of G containing all the vertices of $V(P)$.*

Proof. Assume the opposite. Let P' be an o_{-1}-path containing all the vertices of $V(P)$ such that $\mathrm{def}(P')$ is as small as possible. Then $\mathrm{def}(P') \geq 1$. Without loss of generality, we assume that $P' = v_1v_2\cdots v_p$ and $v_kv_{k+1} \notin E(G)$ and $d(v_k) + d(v_{k+1}) \geq n-1$, where $1 \leq k \leq p-1$.

If v_k and v_{k+1} have a common neighbor in $V(G)\backslash V(P)$, denote it by x. Then $P'' = P'[v_1, v_k]v_kxv_{k+1}P'[v_{k+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P)$ with deficit smaller than $\mathrm{def}(P')$, a contradiction.

So we assume that $N_{G-P'}(v_1)\cap N_{G-P'}(v_k) = \emptyset$. Then $d_{P'}(v_k)+d_{P'}(v_{k+1}) \geq |V(P')| - 1$ since $d(v_k) + d(v_{k+1}) \geq n-1$.

If $v_1v_{k+1} \in E(G)$, then $P'' = P'[v_k, v_1]v_1v_{k+1}P'[v_{k+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P)$ with deficit smaller than def(P'), a contradiction. Thus we assume that $v_1v_{k+1} \notin E(G)$ and similarly, $v_pv_k \notin E(G)$. Thus, there exists an integer $i \in [1, p-1] \setminus \{k\}$ such that $v_i \in N_P(v_k)$ and $v_{i+1} \in N_P(v_{k+1})$.

If $1 \leq i \leq k-1$, then $P'' = P'[v_1, v_i]v_iv_kP'[v_k, v_{i+1}]\ v_{i+1}v_{k+1}P'[v_{k+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P)$ with deficit smaller than def(P'), a contradiction. If $k+1 \leq i \leq p-1$, then $P'' = P'[v_1, v_k]v_kv_iP'[v_i, v_{k+1}]$ $v_{k+1}v_{i+1}P'[v_{i+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P)$ with deficit smaller than def(P'), a contradiction. This completes the proof of Lemma 1. □

In the following, we use $\widetilde{E}_{-1}(G)$ to denote the set $\{uv : uv \in E(G)$ or $d(u) + d(v) \geq n-1\}$.

The next lemma provides some structure on claw-o_{-1}-heavy graphs that contain a cut vertex.

Lemma 2. *Let G be a connected claw-o_{-1}-heavy graphs and let x be a cut vertex of G. Then*

(1) *$G - x$ contains exactly two components; and*

(2) *if x_1 and x_2 are two neighbors of x in a common component, then $x_1x_2 \in \widetilde{E}_{-1}(G)$.*

Proof. If there are at least three components of $G - x$, then let H_1, H_2 and H_3 be three of these components. Let x_1, x_2 and x_3 be neighbors of x in H_1, H_2 and H_3, respectively. Then the subgraph induced by $\{x, x_1, x_2, x_3\}$ is a claw. Moreover, for $1 \leq i < j \leq 3$, $d(x_i) + d(x_j) \leq |V(H_i)| + |V(H_j)| \leq n-2$, a contradiction. Thus, $G - x$ has exactly two components.

Let x_1 and x_2 be two neighbors of x in a common component H. If $x_1x_2 \notin E(G)$, then let x' be a neighbor of x in the other component H'. Then the subgraph induced by $\{x, x_1, x_2, x'\}$ is a claw. Moreover, for $i = 1, 2$, $d(x_i) + d(x') \leq |V(H)| - 1 + |V(H')| \leq n-2$. Since G is claw-o_{-1}-heavy, we get $d(x_1) + d(x_2) \geq n-1$. This completes the proof of Lemma 2. □

3.4 Proof of Theorem 3.3

Let G be a connected P_3-o_{-1}-heavy graph on n vertices, and let $P = v_1v_2\cdots v_p$ be a longest path of G. Assume that G is not traceable. Then $V(G)\backslash V(P) \neq \emptyset$. Since G is connected, there exists a vertex $x \in V(G)\backslash V(P)$ adjacent to P. Let v_i be a neighbor of x in P. Clearly $v_i \neq v_p$; otherwise $P' = Pv_px$ is a longer path than P. If $xv_{i+1} \in E(G)$, then $P' = P[v_1, v_i]v_ixv_{i+1}P[v_{i+1}, v_p]$ is a longer path than P, a contradiction. Thus we assume that $xv_{i+1} \notin E(G)$. Since G is P_3-o_{-1}-heavy, $d(x) + d(v_{i+1}) \geq n - 1$. Thus $P' = P[v_1, v_i]v_ixv_{i+1}P[v_{i+1}, v_p]$ is an o_{-1}-path of G. By Lemma 2, there is a path of G containing all the vertices of P', a contradiction. This completes the proof of Theorem 3.3.

3.5 Proof of Theorem 3.5

Let G be a connected claw-o_{-1}-heavy and Z_1-free graph on n vertices. We are going to prove that G is traceable. If $n = 1$ or $n = 2$, then the result is trivially true. So we assume that $n \geq 3$. We distinguish two cases.

Case 1. G is separable.

If G itself is a path, then there is nothing to prove. Thus we assume that G is not a path. Hence there must be a cut vertex of G with degree at least 3. Let x be such a cut vertex. By Lemma 1, $G - x$ has exactly two components. Let C and D be the two components of $G - x$. Since $d(x) \geq 3$, without loss of generality, we assume that x has at least two neighbors in D.

If x is contained in a triangle $xx'x''$, then x' and x'' are in a common component of $G - x$. Without loss of generality, Let $x', x'' \in V(D)$. Let w be a neighbor of x in C. Then the subgraph induced by $\{x, x', x'', w\}$ is a Z_1, a contradiction. Hence we assume that x is not contained in a triangle and thus that $N(x)$ is an independent set.

Let y be a neighbor of x. If y is contained in a triangle $yy'y''$, then clearly $xy', xy'' \notin E(G)$; otherwise x will be contained in a triangle. Thus the subgraph induced by $\{y, y', y'', x\}$ is a Z_1, a contradiction. Hence we assume that y is not contained in a triangle and that $N(y)$ is an independent set. Similarly, let z be a vertex at distance 2 from x, and let y be a common neighbor of

x and z. If z is contained in a triangle $zz'z''$, then clearly $yz', yz'' \notin E(G)$; otherwise y will be contained in a triangle. Thus the subgraph induced by $\{z, z', z'', y\}$ is a Z_1, a contradiction. Hence we assume that z is not contained in a triangle and that $N(z)$ is an independent set. We conclude that every vertex adjacent to x or at distance 2 from x is contained in no triangles.

Let w be a neighbor of x in C, and let y be a neighbor of x in D. Let y' be a neighbor of x in D other than y. Since $yy' \notin E(G)$, by Lemma 1, we get that $d(y) + d(y') \geq n - 1$. Without loss of generality, we assume that $d(y) \geq (n-1)/2$. Noting that x and y have no common neighbors, we get that $d(x) \leq (n+1)/2$. We distinguish three cases according to the degree of x.

Case A. $d(x) = (n+1)/2$.

In this case, n is odd. Let $Y = N(x) \setminus \{w\}$ and $Z = V(G) \setminus Y \setminus \{x, w\}$. Then $|Y| = (n-1)/2$ and $|Z| = (n-3)/2$. Since $d(y) \geq (n-1)/2$ and y is not adjacent to any vertices in $Y \cup \{w\}$, y is adjacent to every vertex in Z and $d(y) = (n-1)/2$. This implies that $Z \subset V(D)$. Thus every vertex in $N_C(x)$ will have degree 1. This implies that there is only one vertex w in C and $Y \subset V(D)$.

Note that $d(y) = (n-1)/2$. Let y' be a vertex in Y other than y. By Lemma 1, $d(y)+d(y') \geq n-1$. Thus $d(y') \geq (n-1)/2$. Since y' is not adjacent to any vertices in $Y \cup \{w\}$, y' is adjacent to every vertex in Z. This implies that every vertex of Y is adjacent to every vertex of Z.

Let $Y = \{y_1, y_2, \ldots, y_{(n-1)/2}\}$ and $Z = \{z_1, z_2, \ldots, z_{(n-3)/2}\}$. Then $P = wxy_1z_1y_2z_2 \cdots\ z_{(n-3)/2}y_{(n-1)/2}$ is a Hamilton path of G, completing the proof in this case.

Case B. $d(x) = n/2$.

In this case, n is even and $d(y) \geq n/2$. Let $Y = N(x) \setminus \{w\}$ and $Z = V(G) \setminus Y \setminus \{x, w\}$. Then $|Y| = (n-2)/2$ and $|Z| = (n-2)/2$. Since $d(y) \geq n/2$ and y is not adjacent to any vertices in $Y \cup \{w\}$, y is adjacent to every vertex in Z and $d(y) = n/2$. This implies that $Z \subset V(D)$. Thus every vertex in $N_C(x)$ will have degree 1, there is only one vertex w in C, and $Y \subset V(D)$. Note that $d(x) \geq 3$, $n \geq 6$ and $|Z| \geq 2$.

Let $Y = \{y_1, y_2, \ldots, y_{(n-2)/2}\}$, where y_1 has the smallest degree of all vertices in Y, and $Z = \{z_1, z_2, \ldots, z_{(n-2)/2}\}$, where z_1 has the largest degree of all vertices in Z. For every vertex y_i in Y other than y_1, since $d(y_1)+d(y_i) \geq n-1$, we have $d(y_i) \geq n/2$. Since y_i is not adjacent to any vertices in $Y \cup \{w\}$, y_i is adjacent to every vertex of Z. This implies that every vertex of $Y \setminus \{y_1\}$ is adjacent to every vertex of Z.

Let z_i be a vertex of Z other than z_1. Then the subgraph induced by $\{y, x, z_1, z_i\}$ is a claw. Since $d(x) = n/2$, $d(z_1) \geq (n-2)/2$. Noting that z_1 is not adjacent to any vertices in $Z \cup \{x, w\}$, z_1 is adjacent to every vertex in Y and $y_1 z_1 \in E(G)$.

Thus $P = wxy_1z_1y_2z_2 \cdots\ y_{(n-2)/2}z_{(n-2)/2}$ is a Hamilton path of G, completing the proof in this case.

Case C. $d(x) \leq (n-1)/2$.

Note that $d(x) \geq 3$, $n \geq 7$ and $d(y) \geq (n-1)/2 \geq 3$. Let z be a neighbor of y other than x with the largest degree. Let z' be a neighbor of y other than x and z. Then the subgraph induced by $\{y, x, z, z'\}$ is a claw. Since $d(x) \leq (n-1)/2$, $d(z) \geq (n-1)/2$.

Let $Y = N(z)$ and $Z = V(G) \setminus Y \setminus \{x, w\}$. Note that $d(y) \geq (n-1)/2$ and y is not adjacent to any vertices in $Y \cup \{w\}$, and that $d(z) \geq (n-1)/2$ and z is not adjacent to any vertices in $Z \cup \{x, w\}$. This implies that $|Y| = (n-1)/2$, $|Z| = (n-3)/2$, and y is adjacent to every vertex in Z. Hence there is only one vertex w in C, and $Y, Z \subset V(D)$.

Note that x has at least two neighbors in Y. Let $Y = \{y_1, y_2, \ldots, y_{(n-1)/2}\}$, where y_1 and y_2 are two neighbors of x, and let $Z = \{z_1, z_2, \ldots, z_{(n-3)/2}\}$, where z_1 is has the smallest degree of all the vertices in Z. Since $d(y_1)+d(y_2) \geq n-1$ and y_1 and y_2 are not adjacent to any vertices in $Y \cup \{w\}$, y_1 and y_2 are adjacent to all vertices in Z, and then $y_1z_1, y_2z_1 \in E(G)$.

Let z_i be a vertex of Z other than z_1. Then the subgraph induced by $\{y, x, z_1, z_i\}$ is a claw. Since $d(x) \leq (n-1)/2$, $d(z_i) \geq (n-1)/2$. Noting that z_i is not adjacent to any vertices in $Z \cup \{x, w\}$, we get that z_i is adjacent to every vertex in Y. This implies that every vertex of Y is adjacent to every vertex of $Z \setminus \{z_1\}$.

Thus $P = wxy_1z_1y_2z_2 \cdots\ z_{(n-3)/2}y_{(n-1)/2}$ is a Hamilton path of G. This completes the proof for Case 1.

Case 2. G is 2-connected.

Let $P = v_1v_2 \cdots v_p$ be a longest path of G. Assume that G is not traceable. Then $V(G)\backslash V(P) \neq \emptyset$. Since G is 2-connected, there exists a path R with two end vertices in P and of length at least 2 that is internally-disjoint with P. Let $R = x_0x_1x_2 \cdots x_{r+1}$, where $x_0 = v_i$ and $x_{r+1} = v_j$. Clearly $i \neq 1, p$ and $j \neq 1, p$. Without loss of generality, we assume that $2 \leq i < j \leq p-1$. We prove four claims to complete the proof for Case 2.

Claim 1. Let $x \in V(R) \setminus \{v_i, v_j\}$ and $y \in \{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\}$. Then $xy \notin \widetilde{E}_{-1}(G)$.

Proof. Without loss of generality, we assume $y = v_{i-1}$. If $xv_{i-1} \in \widetilde{E}_{-1}(G)$, then $P' = P[v_1, v_{i-1}]v_{i-1}xR[x, v_i]\ v_iP[v_i, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R[x, v_i])$. By Lemma 2, there is a path containing all the vertices of P', a contradiction. □

Claim 2. $v_{i-1}v_{i+1} \in \widetilde{E}_{-1}(G)$ and $v_{j-1}v_{j+1} \in \widetilde{E}_{-1}(G)$.

Proof. If $v_{i-1}v_{i+1} \notin E(G)$, by Claim 1, the graph induced by $\{v_i, x_1, v_{i-1}, v_{i+1}\}$ is a claw, where $d(x_1) + d(u_{i\pm1}) < n-1$. Since G is a claw-o_{-1}-heavy graph, we get that $d(v_{i-1}) + d(v_{i+1}) \geq n$.

The second assertion can be proved similarly. □

Claim 3. $v_{i-1}v_{j-1} \notin \widetilde{E}_{-1}(G)$ and $v_{i+1}v_{j+1} \notin \widetilde{E}_{-1}(G)$.

Proof. If $v_{i-1}v_{j-1} \in \widetilde{E}_{-1}(G)$, then $P' = P[v_1, v_{i-1}]v_{i-1}v_{j-1}P[v_{j-1}, v_i]v_iRv_j$ $P[v_j, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, a contradiction.

The second assertion can be proved similarly. □

Claim 4. Either $v_{i-1}v_{i+1} \in E(G)$ or $v_{j-1}v_{j+1} \in E(G)$.

Proof. Assume the opposite. By Claim 2, $d(v_{i-1}) + d(v_{i+1}) \geq n-1$ and $d(v_{j-1}) + d(v_{j+1}) \geq n-1$. By Claim 3, $d(v_{i-1}) + d(v_{j-1}) < n-1$ and $d(v_{i+1}) + d(v_{j+1}) < n-1$, a contradiction. □

Without loss of generality, we assume that $v_{i-1}v_{i+1} \in E(G)$. Then the subgraph induced by $\{v_i, v_{i-1}, v_{i+1}, x_1\}$ is a Z_1, a contradiction.

This completes the proof of Theorem 3.5.

3.6 Proof of Theorem 3.6

Let G be a connected claw-o_{-1}-heavy and P_4-free graph on n vertices. We are going to prove that G is traceable. If $n = 1$ or $n = 2$, the result is trivially true, so we assume that $n \geq 3$. We distinguish two cases.

Case 1. G is separable.

Let x be a cut vertex of G. By Lemma 1, $G-x$ has exactly two components. Let C and D be the two components of $G - x$.

If there is a vertex in D which is not adjacent to x, then let z be a vertex in D with distance 2 from x, and let y be a common neighbor of x and z. Let w be a neighbor of x in C. Then $wxyz$ is an induced P_4 of G, a contradiction. Thus x is adjacent to every vertex in D. By Lemma 1, for every two vertices y and y' in D, $yy' \in \widetilde{E}_{-1}(G)$. Similarly, x is adjacent to every vertex in C, and for every two vertices w and w' in C, $ww' \in \widetilde{E}_{-1}(G)$.

Let $V(C) = \{w_1, w_2, \ldots, w_k\}$ and $V(D) = \{y_1, y_2, \ldots, y_l\}$, where $k+l+1 = n$. Then $P' = w_1w_2 \cdots w_kxy_1y_2 \cdots y_l$ is an o_{-1}-path of G. By Lemma 2, there is a path P containing all the vertices of P', which is a Hamilton path. This completes the proof for Case 1.

Case 2. G is 2-connected.

Let $P = v_1v_2 \cdots v_p$ be a longest path of G. Assume that G is not traceable. Then $V(G) \backslash V(P) \neq \emptyset$. Since G is 2-connected, there exists a path R with two end vertices in P and of length at least 2 which is internally-disjoint with P. Let $R = x_0x_1x_2 \cdots x_{r+1}$, where $x_0 = v_i$ and $x_{r+1} = v_j$. Clearly, $i \neq 1, p$ and $j \neq 1, p$. Without loss of generality, we assume that $2 \leq i < j \leq p - 1$.

Similarly as in Section 3.5, we can prove a number of claims, as follows.

Claim 1. Let $x \in V(R) \setminus \{v_i, v_j\}$ and $y \in \{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\}$. Then $xy \notin \widetilde{E}_{-1}(G)$.

Claim 2. $v_{i-1}v_{i+1} \in \widetilde{E}_{-1}(G)$ and $v_{j-1}v_{j+1} \in \widetilde{E}_{-1}(G)$.

Next we prove two more claims.

Claim 3. $v_iv_{j-1} \notin E(G)$.

Proof. If $v_iv_{j-1} \in E(G)$, then $P' = P[v_1, v_{i-1}]v_{i-1}v_{i+1}P[v_{i+1}, v_{j-1}]v_{j-1}v_iRv_j$ $P[v_j, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, a contradiction. □

Let v_k be the first vertex in $P[v_{i+1}, v_{j-1}]$ that is not adjacent to v_i. We have that $i + 2 \leq k \leq j - 1$.

Claim 4. $x_1v_{k-1} \notin E(G)$ and $x_1v_k \notin E(G)$.

Proof. If $v_{k-1} = v_{i+1}$, then by Claim 1, $x_1v_{i+1} \notin E(G)$. If $i + 2 \leq k - 1 \leq j - 2$ and $x_1v_{k-1} \in E(G)$, then $P' = P[v_1, v_{i-1}]v_{i-1}v_{i+1}P[v_{i+1}, v_{k-2}]v_{k-2}v_i$ $x_1v_{k-1}P[v_{k-1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, a contradiction. Thus $x_1v_{k-1} \notin E(G)$.

If $z_1v_k \in E(G)$, then $P' = P[v_1, v_{i-1}]v_{i-1}v_{i+1}P[v_{i+1}, v_{k-1}]v_{k-1}v_ix_1v_kP[v_k,$ $v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, a contradiction. Thus $x_1v_k \notin E(G)$. □

Now $x_1v_iv_{k-1}v_k$ is an induced P_4, a contradiction.

This completes the proof of Theorem 3.6.

3.7 Remarks

In this short section, we will explain why we use the concept of o_{-1}-heavy subgraphs in this chapter. Firstly, consider the non-traceable complete bipartite graph $K_{k,k+2}$. Note that every subgraph of $K_{k,k+2}$ (other than K_1 and K_2) is o_{-2}-heavy. Thus, if we consider o_{-2}-heavy subgraph conditions, we cannot get any reasonable subgraph conditions for guaranteeing traceability.

Next, we consider o_r-heavy subgraph conditions for $r \geq 0$. We will show that we cannot get better subgraph conditions for traceability other than those that have been used in Theorems 3.4 and 3.7.

Theorem 3.8. *Let $r \geq 0$ be an integer. Let R and S be connected graphs with $R, S \neq P_3$, and let G be a connected graph. Then G being $\{R, S\}$-o_r-heavy implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$.*

Theorem 3.9. *Let $r \geq 0$ be an integer. Let S be connected graphs with $S \neq P_3$, and let G be a connected claw-o_r-heavy graph. Then G being S-free implies G is traceable if and only if $S = C_3$, Z_1 or P_4.*

The ‘if’ parts of these two theorems can be deduced from Theorems 3.4 and 3.7 directly in a straightforward way. Here we indicate how to prove the ‘only-if’ parts of these results. In Figure 3.2, take $k \geq 3$ and $n \geq 4k + r - 2$ in G_1, and take $k \geq r + 5$ and $n \geq 6k + r + 10$ in G_2. Then G_1 is $\{K_{1,3}, P_4\}$-o_r-heavy and G_2 is $\{K_{1,3}, Z_1\}$-o_r-heavy. Since the two graph are non-traceable, we can use them in an obvious way to prove the ‘only-if’ parts of Theorems 3.8 and 3.9.

Chapter 4

Forbidden pairs for traceability of block-chains

4.1 Introduction

A graph is traceable if it contains a Hamilton path , i.e., a path containing all its vertices. If a graph is connected and P_3-free, then it is a complete graph and it is trivially traceable. In fact, it is not difficult to show that P_3 is the only single subgraph H such that every connected H-free graph is traceable. Moving to the more interesting case of pairs of subgraphs, the following theorem on forbidden pairs for traceability is well-known.

Theorem 4.1 (Duffus, Gould and Jacobson [21]). *If G is a connected $\{K_{1,3}, N\}$-free graph, then G is traceable.*

Obviously, if H is an induced subgraph of N, then the pair $\{K_{1,3}, H\}$ is also a forbidden pair that guarantees the traceability of every connected graph. In fact, Faudree and Gould proved that these are the only forbidden pairs with this property.

Theorem 4.2 (Faudree and Gould [24]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$, P_4, Z_1, B or N.*

Adopting the terminology of [26], we say that a graph is a *block-chain* if it is nonseparable (2-connected or P_1 or P_2) or it has at least one cut vertex and exactly two end blocks. Note that every traceable graph is necessarily a block-chain, but that the reverse does not hold in general. Also note that it is easy to check by a polynomial algorithm whether a given graph is a block-chain or not. In the 'only-if' part of the proof of Theorem 4.2 many graphs are used that are not block-chains (and are therefore trivially non-traceable). A natural extension is to consider forbidden subgraph conditions for a block-chain to be traceable. In this chapter, we characterize all the pairs of subgraphs with this property. First note that, similarly as in the above analysis, it is easy to check that any P_3-free block-chain is traceable. We will show that P_3 is the only single forbidden subgraph with this property.

Theorem 4.3. *The only connected graph S such that a block-chain being S-free implies it is traceable is P_3.*

Next we will prove the following characterization of all pairs of connected graphs R and S other than P_3 guaranteeing that every $\{R, S\}$-free block-chain is traceable.

Theorem 4.4. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a block-chain. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,3}$, or $R = K_{1,4}$ and $S = P_4$.*

It is interesting to note that one of the pairs does not include the claw, in contrast to all existing characterizations of pairs of forbidden subgraphs for hamiltonian properties we encountered.

In Section 4.2, we prove the 'only if' part of Theorems 4.3 and 4.4. For the 'if' part of Theorem 4.4, it is sufficient to prove the following results.

Theorem 4.5. *If G is a $\{K_{1,4}, P_4\}$-free block-chain, then G is traceable.*

Theorem 4.6. *If G is a $\{K_{1,3}, N_{1,1,3}\}$-free block-chain, then G is traceable.*

We prove Theorems 4.5 and 4.6 in Sections 4.4 and 4.5, respectively.

4.2 The 'only-if' part of Theorems 4.3 and 4.4

We first sketch some families of graphs that are block-chains but not traceable (see Figure 4.1). When we say that a graph is of *type* G_i we mean that it is

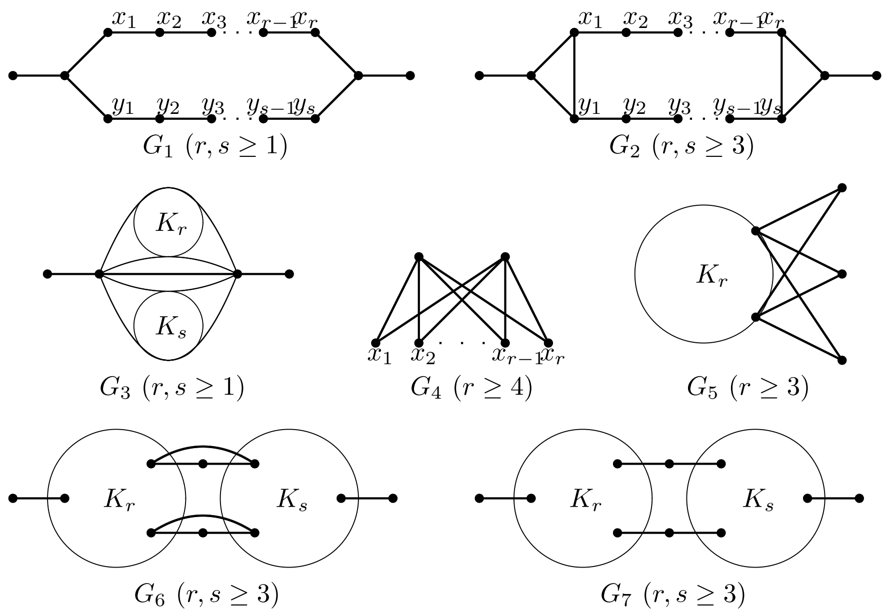


Figure 4.1: Some block-chains that are not traceable

one particular, but arbitrarily chosen member of the family indicated by G_i in Figure 4.1.

If S is a connected graph such that every S-free block-chain is traceable, then S must be a common induced subgraph of all graphs of type G_1, G_2 and G_4. Note that the only largest common induced connected subgraph of graphs of type G_1, G_2 and G_4 is a P_3, so we have $S = P_3$. This completes the proof of the 'only-if' part of the statement of Theorem 4.3.

Let R and S be two connected graphs other than P_3 such that every $\{R, S\}$-free block-chain is traceable. Then R or S must be an induced subgraph of all graphs of type G_1. Without loss of generality, we assume that R is an induced subgraph of all graphs of type G_1. If $R \neq K_{1,3}$, then R must contain an induced P_4. Note that the graphs of type G_4 and G_5 are all P_4-free, so they must contain S as an induced subgraph. Since the only common induced connected subgraph of the graphs of type G_3 and G_4 other than P_3 is $K_{1,3}$ or

$K_{1,4}$, we have that $S = K_{1,3}$ or $K_{1,4}$. This implies that R or S must be $K_{1,3}$ or $K_{1,4}$. Without loss of generality, we assume that $R = K_{1,3}$ or $K_{1,4}$.

Suppose first that $R = K_{1,4}$. Noting that the graphs of type G_1, G_2 and G_3 are all $K_{1,4}$-free, S must be a common induced subgraph of the graphs of type G_1, G_2 and G_3. Since the only common induced connected subgraph of the graphs of type G_1, G_2 and G_3 other than P_3 is P_4, we have $S = P_4$.

Suppose now that $R = K_{1,3}$. Note that the graphs of type G_2 are claw-free. So S must be an induced subgraph of all graphs of type G_2. The common induced connected subgraphs of such graphs have the form P_i, Z_i, $B_{i,j}$ or $N_{i,j,k}$. Note that graphs of type G_6 are claw-free and do not contain an induced P_7 or Z_4, and that graphs of type G_7 are claw-free and do not contain an induced $B_{2,2}$. So R must be an induced connected subgraph of P_6, Z_3, $B_{1,3}$ or $N_{1,1,3}$. Since P_6, Z_3 and $B_{1,3}$ are induced subgraphs of $N_{1,1,3}$, R must be an induced connected subgraph of $N_{1,1,3}$. This completes the proof of the 'only-if' part of the statement of Theorem 4.4.

4.3 Some preliminaries

Let G be a graph. For a subgraph B of G, when no confusion can occur, we also use B to denote its vertex set; similarly, for a subset C of $V(G)$, we also use C to denote the subgraph induced by C.

For a graph G, we use $\kappa(G)$ to denote the connectivity of G and $\alpha(G)$ to denote the *independence number* of G, i.e., the maximum number of vertices no two of which are adjacent. The following theorem on hamiltonian and traceable graphs is well-known and will be used in the sequel.

Theorem 4.7 (Chvátal and Erdös [19]). *Let G be a graph on at least three vertices. If $\alpha(G) \leq \kappa(G)$, then G is hamiltonian. If $\alpha(G) \leq \kappa(G) + 1$, then G is traceable.*

We will also repeatedly use the following structural lemmas on claw-free graphs.

Lemma 1. *If G is a connected claw-free graph, and x is a cut vertex of G, then*

(1) *$G - x$ has exactly two components; and*

(2) *if* x_1, x_2 *are two neighbors of* x *in a common component, then* $x_1x_2 \in E(G)$.

Proof. Note that for every component H of $G - \{x, y\}$, H must contain a neighbor of x. If there are at least three components of $G-x$, then let H_1, H_2 and H_3 be three components. Let x_1, x_2 and x_3 be neighbors of x in H_1, H_2 and H_3, respectively. Then the subgraph induced by $\{x, x_1, x_2, x_3\}$ is a claw, a contradiction. Thus $G - x$ has exactly two components.

Let x_1, x_2 be two neighbors of x in a common component. If $x_1x_2 \notin E(G)$, then let x' be a neighbor of x in the other component. Then the subgraph induced by $\{x, x_1, x_2, x'\}$ is a claw, a contradiction. Thus $x_1x_2 \in E(G)$. □

Lemma 2. *If* G *is a 2-connected claw-free graph, and* $\{x, y\}$ *is a vertex cut of* G*, then*

(1) $G - \{x, y\}$ *has exactly two components; and*

(2) *if* x_1, x_2 *are two neighbors of* x *in a common component, then* $x_1x_2 \in E(G)$.

This lemma can be proved by using Lemma 1 on $G - y$.

In the following, by the concept *cut* we always refer to a vertex cut with 2 vertices.

A graph G is said to be *homogeneously traceable*, if for every vertex x of G, there is a Hamilton path starting from x. We will use the following recent theorem on homogeneously traceable graphs (see Chapter 7).

Theorem 4.8. *Let* R *and* S *be connected graphs with* $R, S \neq P_3$ *and let* G *be a 2-connected graph. Then* G *being* $\{R, S\}$*-free implies* G *is homogeneously traceable, if and only if (up to symmetry)* $R = K_{1,3}$ *and* S *is an induced subgraph of* $B_{1,4}$, $B_{2,3}$ *or* $N_{1,1,3}$.

Let G be a graph, let B and C be two subgraphs of G (possibly non-disjoint), and let H be a subgraph of G which is disjoint from $B \cup C$. If P is a path with its two end-vertices $x \in V(B)$ and $y \in V(C)$ and its internal vertex set $V(P) \setminus \{x, y\} = V(H)$, then we call P *a perfect path of* H *to* B *and* C *(in* G*)* ; if $B = C$, then we also call P *a perfect path of* H *to* B *(in* G*)*. If there is a perfect path of H to B (and C), then we say H *supports* a perfect path to B (and C).

4.4 Proof of Theorem 4.5

Let G be a $\{K_{1,4}, P_4\}$-free block-chain. We are going to prove that G is traceable.

If G contains only one or two vertices, then it is trivially traceable. So we assume that G has at least three vertices. If G is complete, then the result is trivially true. So we assume that G is not complete. Let X be a minimum vertex cut of G.

Clearly each vertex of X has a neighbor in each component of $G-X$. Now we claim that each vertex of X is adjacent to every vertex in $G-X$. Let $x \in X$ and $y \in H$, where H is a component of $G-X$. If $xy \notin E(G)$, then let Q be a shortest path from x to y with all internal vertices in H. Let H' be a component of $G-X$ other than H and let y' be a neighbor of x in H'. Then Qxy' is an induced path with at least four vertices. This contradicts that G is P_4-free. Thus as we claimed, each vertex of X is adjacent to every vertex in $G-X$.

Let S be an independent set of G. Then S is either contained in X or in $G-X$. Without loss of generality, we assume that $S \subset G-X$. Let x be a vertex of X. If S has at least four vertices, then the subgraph induced by $\{x\} \cup S$ is a $K_{1,t}$ with $t \geq 4$, contradicting that G is $K_{1,4}$-free. This implies that the independence number $\alpha(G) \leq 3$.

If G is 2-connected, then $\alpha(G) \leq \kappa(G)+1$. By Theorem 4.7, G is traceable. So we assume that G has a cut vertex.

Let x be a cut vertex of G. Let H be an arbitrary component of $G-x$. We claim that there is a Hamilton path of $H \cup \{x\}$ starting from x. If H contains only one vertex, then the result is trivially true. So we assume that H has at least two vertices. Note that H is connected and x is adjacent to every vertex of H. Hence $H \cup \{x\}$ is 2-connected. If H contains an independent set S with three vertices, then let y be a neighbor of x in H', where H' is a component of $G-x$ other than H. Then the subgraph induced by $\{x, y\} \cup S$ is a $K_{1,4}$, a contradiction. This implies that $\alpha(H \cup \{x\}) \leq 2$. By Theorem 4.7, $H \cup \{x\}$ is hamiltonian. Thus it contains a Hamilton path starting from x.

It is not difficult to see that either $H \cup \{x\}$ is an end block or H contains an end block of G. If $G-x$ has at least three components, then there will be at least three end blocks of G, contradicting that G is a block-chain. Thus $G-x$ has exactly two components. Let H_1 and H_2 be the two components of $G-x$. Let Q_i, $i=1,2$, be the Hamilton path of $H_i \cup \{x\}$ starting from x. Then Q_1xQ_2 is a Hamilton path of G. This completes the proof of Theorem 4.5.

4.5 Proof of Theorem 4.6

Let G be a $\{K_{1,3}, N_{1,1,3}\}$-free block-chain. We are going to prove that G is traceable.

We use induction on $|V(G)|$. If G contains only one or two vertices, then the result is trivially true. So we assume that G contains at least three vertices.

If G is 2-connected, then by Theorem 4.8, G is (homogeneously) traceable. Thus we assume that G has at least one cut vertex. We also assume that G is non-traceable, and will reach a contradiction in all cases.

Claim 1. If x is a cut vertex of G, then at least one of the components of $G-x$ consists of one isolated vertex.

Proof. By Lemma 1, there are exactly two components in $G-x$. Let H_1 and H_2 be the two components. Suppose that both H_1 and H_2 have at least 2 vertices. For $i=1,2$, let y_i be a neighbor of x in H_i, and let G_i be the subgraph of G induced by $H_i \cup \{x, y_{3-i}\}$. It is not difficult to see that G_i is a block-chain, and that y_{3-i} has only one neighbor x in G_i. By the induction hypothesis, there is a Hamilton path Q_i of G_i (starting from y_{3-i}). Then $Q_i' = Q_i - xy_{3-i}$ is a Hamilton path of $H_i \cup \{x\}$ starting from x. Thus $Q_1'xQ_2'$ will be a Hamilton path of G, a contradiction. □

Let x be a cut vertex of G, and let y be an isolated vertex of $G-x$. Clearly the subgraph induced by $\{x, y\}$ is an end block of G. If G has at least three cut vertices, then there will be at least three end blocks of G, a contradiction. Thus we assume that there are at most two cut vertices in G.

Suppose first that there is only one cut vertex in G, and denote it by x. Let y be an isolated vertex of $G-x$, and let H be the component of $G-x$ not containing y. We claim that there is a Hamilton path of $H \cup \{x\}$ starting from x. If H has only one vertex, the result is trivially true. So we assume that H has at least two vertices. If $H \cup \{x\}$ has a cut vertex (note that x is not a cut vertex of $H \cup \{x\}$), then it is also a cut vertex of G, a contradiction. So we assume that $H \cup \{x\}$ is 2-connected. By Theorem 4.8, $H \cup \{x\}$ is homogeneously traceable. Thus as we claimed, there is a Hamilton path Q of $H \cup \{x\}$ starting from x. So yxQ is a Hamilton path of G. This contradiction shows that G has exactly two cut vertices.

Let r and s be the two cut vertices of G, and let r_0 and s_0 be the isolated vertices of $G - r$ and $G - s$, respectively. Let $B = G - \{r_0, s_0\}$. If B has only two vertices r and s, then clearly G is traceable. So we assume that B has at least one vertex other than r and s. Note that if B has a cut vertex, then it is also a cut vertex of G (Clearly r and s are not cut vertices of B), a contradiction. So we assume that B is 2-connected, and it is sufficient to prove that there is a Hamilton path in B from r to s.

Let G_0 be the graph obtained from G by adding an edge r_0s_0. Since G is claw-free and r_0s_0 cannot be an edge of a claw, we see that G_0 is claw-free. Clearly, G_0 is 2-connected. Now we prove the following claims.

Claim 2. If x is a vertex of $B\backslash\{r, s\}$, then $G - x$ is a block-chain if and only if $G_0 - x$ is 2-connected.

Proof. Note that the subgraphs induced by $\{r, r_0\}$ and $\{s, s_0\}$ are two end blocks of $G - x$. If $G - x$ has no other end blocks, then $G_0 - x$ is 2-connected. If $G - x$ has a third end block, then it is also an end block of $G_0 - x$ and G_0 cannot be 2-connected. □

Claim 3. Let $x \in B - \{r, s\}$ and $y \in \{r, r_0, s, s_0\}$. Then $\{x, y\}$ is not a cut of G_0.

Proof. By our assumption, B is 2-connected. Thus $B - x$ is connected. This implies that $G_0 - \{x, y\}$ is connected for $y = r_0$ or s_0. Now we suppose without loss of generality $G - \{x, r\}$ is not connected. Clearly r_0, s and s_0 are in a common component of $G_0 - \{x, r\}$. Let H be the component of $G_0 - \{x, r\}$ not containing r_0, s, s_0. By Lemma 1, we can see that $N_B(r)$ is a clique and is contained in H. Thus every path P of G_0 from H to s_0 will either pass through x or pass through r. But if P passes through r, then it will also pass through r_0. This implies that $\{x, r_0\}$ is a cut of G_0, a contradiction. □

Set

$$N_i = \{v \in B - s : d_{B-s}(v, r) = i\}, \text{ and } j = \max\{i : N_i \neq \emptyset\}.$$

Note that $N_0 = \{r\}$ and $N_1 = N_B(r) \setminus \{s\}$. For a vertex $x \in N_i$, we call the integer i the *level* of x.

Let d be the distance between r and s in B, and let Q be a shortest path in B from r to s. Our next claim shows that $d \leq 3$.

Claim 4. $d \leq 3$.

Proof. Suppose that $d \geq 4$. By Lemma 1, N_1 is a clique. Now we first prove a number of subclaims on the structure of N_i for $i \geq 2$.

Claim 4.1. N_2 and N_3 are cliques.

Proof. Clearly, for every i with $1 \leq i \leq d-1$, Q contains exactly one vertex of N_i. Let $Q' = Qss_0$. Suppose that $Q' = rvwxyz\cdots$. Thus $v \in N_1$, $w \in N_2$, $x \in N_3$, $y \in N_4$ (if $d \geq 5$) or $y = s$ (if $d = 4$) and $z \in N_5$ (if $d \geq 6$) or $z = s$ (if $d = 5$) or $z = s_0$ (if $d = 4$).

First we consider N_2. Let w' be an arbitrary vertex in N_2 other than w. We claim that $ww' \in E(G)$. If $ww' \notin E(G)$, then w and w' have no common neighbors in N_1; otherwise, letting v' be a common neighbor of w and w' in N_1, the subgraph induced by $\{v', r, w, w'\}$ is a claw. Besides, we have $wx \notin E(G)$; otherwise the subgraph induced by $\{x, w, w', y\}$ is a claw. Now let v' be a neighbor of w' in N_1. Then $v' \neq v$, $vw', v'w \notin E(G)$ and the subgraph induced by $\{r, r_0, v', w', v, w, x, y\}$ is an $N_{1,1,3}$, a contradiction. So, as we claimed w is adjacent to all other vertices in N_2.

Let w' and w'' be two arbitrary vertices in N_2 other than w. We claim that $w'w'' \in E(G)$. Suppose that $w'w'' \notin E(G)$. If $w'x \in E(G)$, then similarly as in the above analysis, we can prove that w' is adjacent to all other vertices in N_2 and $w'w'' \in E(G)$. So we assume that $w'x \notin E(G)$, and similarly, $w''x \notin E(G)$. Then the subgraph induced by $\{w, w', w'', x\}$ is a claw, a contradiction. Thus, as we claimed, $w'w'' \in E(G)$. This implies that N_2 is a clique.

Similarly, we now consider N_3. Let x' be an arbitrary vertex in N_3 other than x. We claim that $xx' \in E(G)$. If $xx' \notin E(G)$, then x and x' have no common neighbors in N_2 and $x'y \notin E(G)$. Let w' be a neighbor of x' in N_2. Then $w' \neq w$, $wx', w'x \notin E(G)$, and the subgraph induced by $\{v, r, w', x', w, x, y, z\}$ is an $N_{1,1,3}$, a contradiction. So, as we claimed, x is adjacent to all other vertices in N_3.

Let x' and x'' be two arbitrary vertices in N_3 other than x. We claim that $x'x'' \in E(G)$. Suppose that $x'x'' \notin E(G)$. If $x'y \in E(G)$, then similarly as in the above analysis, we can prove that x' is adjacent to all other vertices in N_3 and $x'x'' \in E(G)$. So we assume that $x'y \notin E(G)$, and similarly, $x''y \notin E(G)$. Then the subgraph induced by $\{x, x', x'', y\}$ is a claw, a contradiction. Thus, as we claimed, $x'x'' \in E(G)$. This implies that N_3 is a clique. □

Claim 4.2. For all i with $1 \leq i \leq j$, N_i is a clique.

Proof. We use induction on i. By Lemma 1 and Claim 4.1, N_1, N_2 and N_3 are cliques. So we assume that $4 \leq i \leq j$ and that N_{i-1} is a clique.

Let x and x' be two arbitrary vertices in N_i. We claim that $xx' \in E(G)$. Suppose $xx' \notin E(G)$. Then x and x' have no common neighbors in N_{i-1}. Let w be a neighbor of x in N_{i-1}, let w' be a neighbor of x' in N_{i-1}, and let v be a neighbor of w in N_{i-2}. By the induction hypothesis, $ww' \in E(G)$. We have $vw' \in E(G)$; otherwise the subgraph induced by $\{w, v, w', x\}$ is a claw. Let Q' be a shortest path of B from v to r. Then the subgraph induced by $\{w, x, w', x'\} \cup V(Q') \cup \{r_0\}$ is an $N_{1,1,\ell}$ with $\ell \geq 3$, a contradiction. Thus, as we claimed, $xx' \in E(G)$. This implies N_i is a clique. □

Claim 4.3. There is a neighbor of s in N_j.

Proof. Assume the contrary. Let i be the maximum level of the neighbors of s, where $3 \leq i \leq j-1$. By Lemma 1, $N_B(s)$ is a clique. This implies that every neighbor of s is either in N_i or in N_{i-1}.

Let y be an arbitrary vertex in N_{i+1}. First we assume that y and s have a common neighbor x in N_i. Let w be a neighbor of x in N_{i-1}. Then $ws \in E(G)$; otherwise the subgraph induced by $\{x, w, y, s\}$ is a claw. Let Q' be a shortest path in B from w to r. Then the subgraph induced by $\{x, y, s, s_0\} \cup V(Q') \cup \{r_0\}$ is an $N_{1,1,\ell}$ with $\ell \geq 3$, a contradiction.

Thus we assume that y and s have no common neighbors in N_i. Let x be a neighbor of y in N_i, let x' be a neighbor of s in N_i, and let w be a neighbor of x in N_{i-1}. Then $xs, x'y \notin E(G)$. By Claim 4.2, $xx' \in E(G)$. If $ws \in E(G)$, then let v be a neighbor of w in N_{i-2}. Then the subgraph induced by $\{w, v, x, s\}$ is a claw. Thus we assume that $ws \notin E(G)$. Now we have $wx' \in E(G)$; otherwise the subgraph induced by $\{x, w, x', y\}$ is a claw. Let Q' be a shortest path of B from w to r. Then the subgraph induced by $\{x, y, x', s\} \cup V(Q') \cup \{r_0\}$ is an $N_{1,1,\ell}$ with $\ell \geq 3$, a contradiction. □

If for some i with $1 \leq i \leq j-1$, N_i consists of only one vertex, say x, then x will be a cut vertex of B, a contradiction to our assumption. Thus we assume that $|N_i| \geq 2$ for all i with $1 \leq i \leq j-1$.

Now we construct a Hamilton path of B from r to s as follows. Let x_j be a neighbor of s in N_j. If N_j consists of x_j, then let $Q_j = sx_j$; otherwise, let y_j be a vertex in N_j other than x_j, let R_j be a Hamilton path of N_j from x_j to y_j, and let $Q_j = sx_jR_j$. Then for $i = j-1, j-2, \ldots, 1$, let x_i be a neighbor of y_{i+1} in N_i (where we take $y_j = x_j$ if $|N_j| = 1$), let y_i be a vertex in N_i other than x_i, let R_i be a Hamilton path of N_i from x_i to y_i, and let $Q_i = Q_{i+1}y_{i+1}x_iR_i$. Then Q_1y_1r is a Hamilton path of B from r to s. □

We next show that $rs \in E(G)$.

Claim 5. $rs \in E(G)$.

Proof. Assume, to the contrary, that $rs \notin E(G)$. Then, from the above we get $d = 2$ or $d = 3$. We distinguish the following two cases according to the value of d.

Case A. $d = 2$.

Let $Q = s_1xs_2$. If $G-x$ is a block-chain, then by the induction hypothesis, $G-x$ contains a Hamilton path P' (from r_0 to s_0). Note that r and s are two cut vertices of $G-x$. Thus the subpath R' of P' from r to s is a Hamilton path of $B-x$. Let x' be the neighbor of r on R'. Then $xx' \in E(G)$ and $R = R' - rx' \cup rxx'$ is a Hamilton path of B from r to s, a contradiction. Thus we assume that $G-x$ is not a block-chain, and by Claim 2, $G_0 - x$ has a cut vertex y. Thus $\{x, y\}$ is a cut of G_0.

By Claim 3, $y \in V(B)\backslash\{r, s\}$. Note that r and s are two neighbors of x and $rs \notin E(G_0)$. By Lemma 2, r and s are in distinct components of $G_0 - \{x, y\}$. But r and s are connected in $G_0 - \{x, y\}$ by the path rr_0s_0s, a contradiction.

Case B. $d = 3$.

Let $Q = rxys$. Similarly as in Case A, we can prove that $G_0 - x$ has a cut vertex. We claim that y is a cut vertex of $G_0 - x$; otherwise let y' be a cut vertex of $G_0 - x$ such that $y' \in B \setminus \{r, s, y\}$. Since r and y are two neighbors of x and $ry \notin E(G_0)$, by Lemma 2, r and y are in distinct components of $G_0 - \{x, y'\}$. But r and y are connected in $G_0 - \{x, y'\}$ by the path rr_0s_0sy, a contradiction. Thus we have that y is a cut vertex of $G_0 - x$, and $\{x, y\}$ is a cut of G_0.

Clearly r, r_0, s and s_0 are in a common component of $G_0 - \{x, y\}$. Let H be the component of $G - \{x, y\}$ not containing r. Using Lemma 2, we get that every neighbor of x is either in $H \cup \{y\}$ or in $\{r\} \cup N_B(r)$, and that every neighbor of y is either in $H \cup \{x\}$ or in $\{s\} \cup N_H(s)$. Recall that we assume that B is 2-connected. Let x' be a neighbor of r other than x, and let y' be a neighbor of s other than y.

If there is a vertex in B other than $\{r, s\} \cup N_B(r) \cup N_B(s) \cup H$, then without loss of generality, we assume that z is such a vertex and $zx' \in E(G)$. Then the subgraph induced by $\{r, r_0, x', z, x, y, s, s_0\}$ is an $N_{1,1,3}$, a contradiction. Thus we assume that there are no vertices in B other than $\{r, s\} \cup N_B(r) \cup N_B(s) \cup H$.

Since B is 2-connected, there is an edge between $N_B(r) \setminus \{x\}$ and $N_B(s) \setminus \{y\}$; otherwise x is a cut vertex of B. Without loss of generality, we assume that $x'y' \in E(G)$. Similarly as in the above analysis, we get that $\{x', y'\}$ is a cut of G_0. But since $N_B(r)$ and $N_B(s)$ are cliques and there are no vertices in B other than $\{r, s\} \cup N_B(r) \cup N_B(s) \cup H$, we have that $G_0 - \{x', y'\}$ is connected, a contradiction. □

Using Lemma 1 and Claim 5, we get that $N_B(s) \setminus \{r\} = N_B(r) \setminus \{s\} = N_1$. We can get a lot of structural information on N_j, by the following claim.

Claim 6. $j \leq 3$ and if $j = 3$, then N_3 is P_3-free.

Proof. If $j \geq 4$, then let z be a vertex of N_4, let y be a neighbor of z in N_3, let x be a neighbor of y in N_2, and let w be a neighbor of x in N_1. Then the subgraph induced by $\{r, r_1, s, s_1, w, x, y, z\}$ is an $N_{1,1,3}$, a contradiction. Thus we have $j \leq 3$.

If $yy'y''$ is an induced P_3 in N_3, then let x be a neighbor of y' in N_2, and let w be a neighbor of x in N_1. We have that either $xy \notin E(G)$ or $xy'' \notin E(G)$; otherwise the subgraph induced by $\{x, w, y, y''\}$ is a claw. Without loss of generality, we assume that $xy'' \notin E(G)$. Then the subgraph induced by $\{r, r_1, s, s_1, w, x, y', y''\}$ is an $N_{1,1,3}$, a contradiction. Thus we conclude that N_3 is P_3-free. □

Claim 7. For every vertex $x \in N_1$, there is unique vertex $x' \in N_1 \setminus \{x\}$ such that $\{x, x'\}$ is a cut of G_0.

Proof. First we prove the existence of the claim. Assume that there are not such vertex. Similarly as in the proof of Claim 5, we have that there is a vertex y in $N_2 \cup N_3$ such that $\{x, y\}$ is a cut of G_0. Let H be the component of $G_0 - \{x, y\}$ not containing r, and let Q' be a shortest path from x to y with all internal vertices in H.

Let R be a shortest path in $B\backslash H$ from y to $N_1 \setminus \{x\}$, and let x' be the end vertex of R other than y. Similarly as in the proof of Claim 5, we have that x' is contained in a cut $\{x', y'\}$ of G_0 for some $y' \in V(B) \setminus \{r, s\}$. Let z' be the neighbor of x' on R. By Lemma 2, r and z' are not contained in a common component of $G_0 - \{x', y'\}$. Note that $rxQ \cup R - z'x'$ is a path from r to z' not passing through x'. We have that y' must be a vertex in $V(Q') \cup V(R) \setminus \{x'\}$. By our assumption $y' \neq x$. If $y' \in H \cup \{y\}$, then let H' be the component of $G_0 - \{x', y'\}$ not containing r. Then every neighbor of y will be either in $H \cup \{x\}$ or in $H' \cup \{x'\}$. Hence every path from y to r will pass through either x or x', and $\{x, x'\}$ is a cut of G_0, a contradiction. Thus we have that $y' \in V(R) \setminus \{x', y\}$.

Let T be the subpath of R' from y to y', let H' be the component of $G_0 - \{x', y'\}$ not containing r, and let z' be a neighbor of y' in H'. Then the subgraph induced by $\{r, r_0, s, s_0\} \cup V(R) \cup V(T) \cup \{z'\}$ is an $N_{1,1,\ell}$ with $\ell \geq 3$, a contradiction. Thus we have that there is a vertex $x' \in N_1$ such that $\{x, x'\}$ is a cut of G_0.

Let H be the component of $G_0 - \{x, x'\}$ not containing r. We have that all the neighbors of x in N_2 are in H; otherwise, let y be a neighbor of x in H, and let y' be a neighbor of x in $N_2 \setminus H$. Then the subgraph induced by $\{x, r, y, y'\}$ is a claw. This implies that for any vertex x'' in $N_1 \setminus \{x, x'\}$, the pair $\{x, x''\}$ is not a cut of G_0. □

By Claim 7, we can partition N_1 into pairs such that each pair is a cut of G_0. The next claim shows how we can pick up the vertices of components in paths between the pairs.

Claim 8. Let $\{t, t'\}$ be a cut of G_0 such that $t, t' \in N_1$, and let H be the component of $G_0 - \{t, t'\}$ not containing r. Then there is a perfect path of H to $\{t, t'\}$.

Proof. If $H \cap N_2$ consists of only one vertex x, then by the 2-connectedness of G, $H \cap N_3 = \emptyset$ and $xt, xt' \in E(G)$. Then $R = txt'$ is a perfect path of H to $\{t, t'\}$. Next we assume that $H \cap N_2$ contains at least two vertices. Note that both t and t' are adjacent to some vertices in $H \cap N_2$. We can divide $H \cap N_2$ into two nonempty subsets C and C' such that every vertex in C is adjacent to t, and every vertex in C' is adjacent to t'.

Recall that $j \leq 3$ and, if $j = 3$, then N_3 is P_3-free, so every component of $H \cap N_3$ is a clique.

Claim 8.1. Let D be a component of $H \cap N_3$. If D is joined to C but not to C', then D supports a perfect path to C; if D is joined to C' but not to C, then D supports a perfect path to C'; and if D is joined to both C and C', then D supports a perfect path to C and C'.

Proof. We distinguish three cases.

Case A. D is joined to C but not to C'.

If D contains only one vertex x, then by the 2-connectedness of B, x has at least two neighbors in C. Let w, w' be two neighbors of x in C. Then $R = wxw'$ is a perfect path of D to C.

Now we assume that D contains at least two vertices. By the 2-connectedness of B, D is joined to C by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in D$ and $w, w' \in C$. Let R' be a Hamilton path of D from x to x'. Then $R = wxR'x'w'$ is a perfect path of D to C.

Case B. D is joined to C' but not to C.

This case can be treated in a similar way as Case A.

Case C. D is joined to both C and C'.

If D consists of the vertex x, then x has at least one neighbor in C and in C'. Let w be a neighbor of x in C, and let w' be a neighbor of x in C'. Then $R = wxw'$ is a perfect path of D to C and C'.

Now we assume that D contains at least two vertices. Clearly D is joined to C and C' by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in D$, $w \in C$ and $w' \in C'$. Let R' be a Hamilton path of D from x to x'. Then $R = wxR'x'w'$ is a perfect path of D to C and C'. $\square$

Let $\mathcal{D} = \{D_1, D_2, \ldots, D_k\}$ be the set of components in $H \cap N_3$ that are joined to C but not to C', let R_i $(1 \leq i \leq k)$ be a perfect path of D_i to C, and let x_i, y_i be the two end vertices of R_i; let $\mathcal{D}' = \{D'_1, D'_2, \ldots, D'_{k'}\}$ be the set of components in $H \cap N_3$ that are joined to C' but not to C, let R'_i $(1 \leq i \leq k')$ be a perfect path of D'_i to C', and let x'_i, y'_i be the two end vertices of R'_i; let $\mathcal{D}'' = \{D''_1, D''_2, \ldots, D''_{k''}\}$ be the set of components in $H \cap N_3$ that are joined to both C and C', let R''_i $(1 \leq i \leq k'')$ be a perfect path of D''_i to C and C', and let x''_i, y''_i be the two end vertices of R''_i, where $x''_i \in C$ and $y''_i \in C'$.

We first assume that k'' is odd. If $\mathcal{D} \neq \emptyset$, then let $w = x_1$; otherwise let $w = x''_1$. Let T be a path from t to w passing through all the vertices in $C \setminus \bigcup_{i=1}^{k}\{x_i, y_i\} \setminus \bigcup_{i=1}^{k''}\{x''_i\}$. If $\mathcal{D}' \neq \emptyset$, then let $w' = y'_{k'}$; otherwise let

$w' = y''_{k''}$. Let T' be a path from t' to w' passing through all the vertices in $C' \backslash \bigcup_{i=1}^{k'} \{x'_i, y'_i\} \backslash \bigcup_{i=1}^{k''} \{y''_i\}$. Then $R = Tx_1R_1y_1 \cdots x_kR_ky_kx''_1R''_1y''_1y''_2R''_2x''_2 \cdots x''_{k''}R''_{k''}y''_{k''}x'_1R'_1y'_1 \cdots x'_{k'}R'_{k'}y'_{k'}T'$ is a perfect path of H to $\{t, t'\}$.

Next we assume that k'' is even. If there is an edge joining C to C' such that its two vertices are not the two end vertices of a common perfect path of some component in $\mathcal{D}''$ (we call such an edge a *good edge*), then let zz' be a good edge, where $z \in C$ and $z' \in C'$. Note that z is possibly an end vertex of a perfect path of some component in $\mathcal{D}$ or $\mathcal{D}''$, or that it is not such an end vertex, and that z' is possibly an end vertex of a perfect path of some component in $\mathcal{D}'$ or $\mathcal{D}''$, or that it is not such an end vertex. So there are nine different cases to consider. Here we only discuss two of the cases; for the other cases, a perfect path of H to $\{t, t'\}$ can be found in a similar way.

If z is not an end vertex of a perfect path of some component in $\mathcal{D}$ or $\mathcal{D}''$, and z' is an end vertex of a perfect path of some component in $\mathcal{D}'$, then without loss of generality, we assume that $z' = x'_1$. If $\mathcal{D} \neq \emptyset$, then let $w = x_1$; otherwise, if $\mathcal{D}'' \neq \emptyset$, then let $w = x''_1$; otherwise let $w = z$. Let T be a path from t to w passing through all the vertices in $C \setminus \bigcup_{i=1}^{k} \{x_i, y_i\} \setminus \bigcup_{i=1}^{k''} \{x''_i\} \setminus \{z\}$. Let T' be a path from t' to $y'_{k'}$ passing through all the vertices in $C' \setminus \bigcup_{i=1}^{k'} \{x'_i, y'_i\} \setminus \bigcup_{i=1}^{k''} \{y''_i\}$. Then $R = Tx_1R_1y_1 \cdots x_kR_ky_kx''_1R''_1y''_1y''_2R''_2x''_2 \cdots y''_{k''}R''_{k''}x''_{k''}\ zx'_1R'_1y'_1 \cdots x'_{k'}R'_{k'}y'_{k'}T'$ is a perfect path of H to $\{t, t'\}$.

If both z and z' are end vertices of perfect paths of some components in $\mathcal{D}''$, then note that zz' is a good edge, so these vertices are not the end vertices of a common perfect path. Without loss of generality, we assume that $z = x''_2$ and $z' = y''_1$. If $\mathcal{D} \neq \emptyset$, then let $w = x_1$; otherwise let $w = x''_1$. Let T be a path from t to w passing through all the vertices in $C \setminus \bigcup_{i=1}^{k} \{x_i, y_i\} \setminus \bigcup_{i=1}^{k''} \{x''_i\}$. If $\mathcal{D}' \neq \emptyset$, then let $w' = y'_{k'}$; otherwise let $w' = y''_{k''}$. Let T' be a path from t' to w' passing through all the vertices in $C' \setminus \bigcup_{i=1}^{k'} \{x'_i, y'_i\} \setminus \bigcup_{i=1}^{k''} \{y''_i\}$. Then $R = Tx_1R_1y_1 \cdots x_kR_ky_kx''_1R''_1y''_1x''_2R''_2y''_2 \cdots x''_{k''}R''_{k''}y''_{k''}x'_1R'_1y'_1 \cdots x'_{k'}R'_{k'}y'_{k'}T'$ is a perfect path of H to $\{t, t'\}$.

Next we assume that each edge joining C to C' is not a good edge.

If C is not joined to C', then $\mathcal{D}'' \neq \emptyset$; otherwise t will be a cut vertex of G. If C is joined to C', then we also have $\mathcal{D}'' \neq \emptyset$, since every edge joining C to C' is not good. Recall that we assume that k'' is even, so we have $k'' \geq 2$.

Let R be a shortest path from x''_1 to y''_1 with all internal vertices in D''_1. Note that $x''_1y''_2 \notin E(G)$; otherwise it is a good edge. Moreover, $ty''_2 \notin E(G)$;

otherwise the subgraph induced by $\{t, r, x_1'', y_2''\}$ is a claw. Thus the subgraph induced by $\{r, r_0, s, s_0, t\} \cup V(R) \cup \{y_2''\}$ is an $N_{1,1,\ell}$ with $\ell \geq 3$, a contradiction. $\square$

Let $N_1 = \{x_i, x_i' : 1 \leq i \leq k\}$ such that for every i with $1 \leq i \leq k$, $\{x_i, x_i'\}$ is a cut of G_0. Let H_i be the component of $G_0 - \{x_i, x_i'\}$ not containing r, and let R_i be a perfect path of H_i to $\{x_i, x_i'\}$. Then $R = rx_1R_1x_1' \cdots x_kR_kx_k's$ is a Hamilton path of B from r to s, our final contradiction.

Chapter 5

Heavy pairs for traceability of block-chains

5.1 Introduction

A graph is called *traceable* if it contains a *Hamilton path* , i.e., a path containing all its vertices. For forbidden subgraph conditions for traceability of connected graphs, the following theorems are well-known.

Theorem 5.1 (Faudree and Gould [24]). *The only connected graph S such that every connected S-free graph is traceable is P_3.*

Theorem 5.2 (Faudree and Gould [24]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of N.*

Forbidding pairs of graphs as induced subgraphs might impose such a strong condition on the graphs under consideration that hamiltonian properties are almost trivially obtained. As an example, one easily shows that, apart from paths and cycles, connected $\{K_{1,3}, Z_1\}$-free graphs are only a matching away from complete graphs, i.e., their complements consist of isolated vertices and isolated edges. This is one of the motivations to relax forbidden subgraph conditions to conditions in which the subgraphs are allowed, but where additional conditions are imposed on these subgraphs if they appear. Early examples of this approach in the context of hamiltonicity and pancyclicity date back to the early 1990s [4,12]. The idea to put a minimum degree bound on one or two of the end-vertices of an induced claw has been explored in [11]. Here we follow the ideas and terminology of [17] by putting an Ore-type degree sum condition on at least one pair of nonadjacent vertices in certain induced subgraphs. These degree sum conditions refer to one of the earliest papers in this area, in which Ore proved that a graph G on $n \geq 3$ vertices is hamiltonian if the degree sum of any two nonadjacent vertices of G is at least n. Ore's result implies that a graph on n vertices is traceable if the degree sum of any two nonadjacent vertices is at least $n-1$. A natural way to find common extensions of such degree sum conditions and forbidden subgraph conditions for traceability is to impose that certain pairs of vertices of induced subgraphs have degree sum at least $n-1$. This motivates the following concepts and terminology.

For connected $\mathcal{H}$-o_{-1}-heavy graphs, unfortunately only a small graph and a pair of small graphs can guarantee their traceability, as was shown in Chapter 3.

Theorem 5.3. *The only connected graph S such that every connected S-o_{-1}-heavy graph is traceable is P_3.*

Theorem 5.4. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R,S\}$-o_{-1}-heavy implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$.*

In this paper, we are going to improve the above results by excluding graphs that are more or less trivially non-traceable. Therefore, we focus on graphs that satisfy a simple and easy to verify necessary condition for traceability. Adopting the terminology of [26], we say that a graph is a *block-chain* if it is nonseparable (2-connected or P_1 or P_2) or it has at least one cut vertex and has exactly two end blocks. Note that every traceable graph is necessarily a block-chain, but that the reverse does not hold. Also note that it is easy to check by a polynomial algorithm whether a given graph is a block-chain or not.

In the 'only-if' part of the proof of Theorem 5.4 many graphs are used that are not block-chains (and are therefore trivially non-traceable). A natural extension is to consider forbidden subgraph and o_{-1}-heavy subgraph conditions for a block-chain to be traceable. In Chapter 4, we characterized all the pairs of forbidden subgraphs with this property.

Theorem 5.5. *The only connected graph S such that every S-free block-chain is traceable is P_3.*

Theorem 5.6. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a block-chain. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,3}$, or $R = K_{1,4}$ and $S = P_4$.*

In this chapter, we characterize the pairs of connected graphs R and S other than P_3 guaranteeing that every $\{R, S\}$-o_{-1}-heavy block-chain is traceable. First note that we can easily obtain that the statement 'every H-o_{-1}-heavy block-chain is traceable' only holds if $H = P_3$. This can be deduced from Theorems 5.3 and 5.4. For o_{-1}-heavy pairs of subgraphs, we will prove the following common extension of Theorems 5.4 and 5.6.

Theorem 5.7. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a block-chain. Then G being $\{R, S\}$-o_{-1}-heavy implies G is traceable, if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of W or N.*

In Section 5.2, we prove the 'only if' part of Theorem 5.7. For the 'if' part of Theorem 5.7, it suffices to prove the following two statements.

Theorem 5.8. *If G is a $\{K_{1,3}, W\}$-o_{-1}-heavy block-chain, then G is traceable.*

Theorem 5.9. *If G is a $\{K_{1,3}, N\}$-o_{-1}-heavy block-chain, then G is traceable.*

We prove Theorems 5.8 and 5.9 in Sections 5.4 and 5.5, respectively.

5.2 The 'only if' part of Theorem 5.7

Let R and S be two graphs other than P_3 such that every $\{R, S\}$-o_{-1}-heavy block-chain is traceable. By Theorem 5.6, we have that (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,3}$, or $R = K_{1,4}$ and $S = P_4$.

In Figure 5.1, we sketched some families of block-chains that are not traceable. All members of these families have exactly two cut vertices, two end blocks consisting of K_2's, and one 2-connected non-end block, so all these graphs are obviously block-chains. Since all the graphs of these families have exactly two vertices with degree 1, it is easy to verify that they do not admit a Hamilton path (between these two vertices, because all the other vertices ought to be internal vertices of any Hamilton path). We leave the details for the reader.

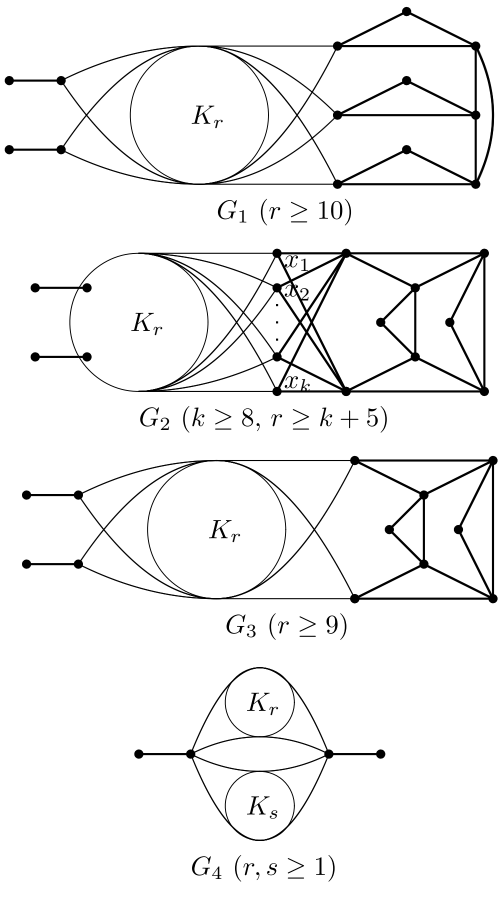


Figure 5.1: Some families of block-chains that are not traceable

Noting that a graph G_4 of type 4 is $\{K_{1,4}, P_4\}$-o_{-1}-heavy, we get that $\{R, S\} \neq \{K_{1,4}, P_4\}$. Thus $R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,3}$.

Note that all graphs G_1, G_2 and G_3 of the first three types are claw-o_{-1}-heavy. So S must be a common induced subgraph of all graphs G_1, G_2 and G_3 that is not o_{-1}-heavy. Note that all graphs G_1 of type 1 are P_6-o_{-1}-heavy, all graphs G_2 of type 2 are Z_3-o_{-1}-heavy, and all graphs G_3 of type 3 are $N_{1,1,2}$-o_{-1}-heavy. The only remaining possibility is that S is an induced subgraph of W or N. This completes the proof of the 'only if' part of the statement of Theorem 5.7.

5.3 Some preliminaries

In the next two sections we will prove Theorems 5.8 and 5.9, respectively. Before we do so, in this section we introduce some additional terminology and notation, and we will prove some useful lemmas.

Let G be a graph and let X be a nonempty subset of $V(G)$. The subgraph of G induced by the set X is denoted by $G[X]$; we use $G - X$ to denote the subgraph induced by $V(G) \setminus X$.

Throughout this chapter, k and ℓ will always denote positive integers. If $k \leq \ell$, we use $[x_k, x_\ell]$ to denote the set $\{x_k, x_{k+1}, \ldots, x_\ell\}$. If $[x_k, x_\ell]$ is a nonempty subset of the vertex set of a graph G, we use $G[x_k, x_\ell]$ instead of $G[[x_k, x_\ell]]$, to denote the subgraph induced by $[x_k, x_\ell]$ in G.

Let P be a path and $x, y \in V(P)$. We use $P[x, y]$ to denote the subpath of P from x to y (inclusive).

Let G be a graph and $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be two pairs of vertices in $V(G)$ with $x_1 \neq x_2$ and $y_1 \neq y_2$. We define an $(\{x_1, x_2\}, \{y_1, y_2\})$-*disjoint path pair*, or briefly an (x_1x_2, y_1y_2)-*pair*, as a union of two vertex-disjoint paths P and Q such that

(1) the origins of P and Q are in $\{x_1, x_2\}$, and

(2) the termini of P and Q are in $\{y_1, y_2\}$.

If G is a graph on $n \geq 2$ vertices, $x \in V(G)$, and a graph G' is obtained from G by adding a new vertex y and a pair of edges yx, yz, where $z \neq x$ is an arbitrary vertex of G, then we say that G' is a *1-extension of G at x to y*. Similarly, if $x_1, x_2 \in V(G)$, $x_1 \neq x_2$, then the graph G' obtained from G by adding two new vertices y_1, y_2 and the edges y_1x_1, y_2x_2 and y_1y_2 is called the *2-extension of* G *at* (x_1, x_2) *to* (y_1, y_2). We also call G' a 1-extension (at x to y) or 2-extension (at (x_1, x_2) to (y_1, y_2)) of G if it contains a spanning subgraph that is a 1-extension (at x to y) or 2-extension (at (x_1, x_2) to (y_1, y_2)) of G.

Let G be a graph and let $u, v, w \in V(G)$ be distinct vertices of G. We say that G is (u, v, w)*-composed* (or briefly *composed*) if G has a spanning subgraph D (called the *carrier* of G) such that there is an ordering $v_{-k}, \ldots, v_0, \ldots, v_\ell$ ($k, \ell \geq 1$) of $V(D)$ (=$V(G)$) and a sequence of graphs $D_1, \ldots, D_r$ ($r \geq 1$) such that

(1) $u = v_{-k}$, $v = v_0$, $w = v_\ell$,
(2) D_1 is a triangle with $V(D_1) = \{v_{-1}, v_0, v_1\}$,
(3) $V(D_i) = [v_{-k_i}, v_{\ell_i}]$ for some k_i, ℓ_i, $1 \leq k_i \leq k$, $1 \leq \ell_i \leq \ell$, and D_{i+1}, $i = 1, \ldots, r-1$, satisfies one of the following:
   (a) D_{i+1} is a 1-extension of D_i at v_{-k_i} to v_{-k_i-1} or at v_{ℓ_i} to v_{ℓ_i+1}, or
   (b) D_{i+1} is a 2-extension of D_i at (v_{-k_i}, v_{ℓ_i}) to $(v_{-k_i-1}, v_{\ell_i+1})$,
(4) $D_r = D$.

The ordering $v_{-k}, \ldots, v_0, \ldots, v_\ell$ will be called a *canonical ordering* and the sequence $D_1, \ldots, D_r$ a *canonical sequence* of D (and also of G). Note that a composed graph G can have several carriers, canonical orderings and canonical sequences. Clearly, a composed graph G and any of its carriers D are 2-connected; moreover, for any canonical ordering, $P = v_{-k} \cdots v_0 \cdots v_\ell$ is a Hamilton path in D (called a *canonical path*), and if $D_1, \ldots, D_r$ is a canonical sequence, then any D_i is $(v_{-k_i}, v_0, v_{\ell_i})$-composed, $i = 1, \ldots, r$. Note that a (u, v, w)-composed graph is also (w, v, u)-composed.

The following lemma on composed graphs will be needed in our proofs. A proof of the lemma can be found in Chapter 6.

Lemma 1. *Let G be a composed graph and let D and $v_{-k}, \ldots, v_0, \ldots, v_\ell$ be a carrier and a canonical ordering of G. Then*
   (1) *D has a Hamilton (v_0, v_{-k})-path, and*
   (2) *for every $v_s \in V(G) \setminus \{v_{-k}\}$, D has a spanning $(v_0 v_\ell, v_s v_{-k})$-pair.*

Let G be a graph on n vertices. A sequence of vertices $v_1 v_2 \cdots v_k$ such that for all $i \in [1, k-1]$, either $v_i v_{i+1} \in E(G)$ or $d(v_i) + d(v_{i+1}) \geq n - 1$, is called an *$o_{-1}$-path* of G.

The following useful lemma on o_{-1}-paths is proved in Chapter 3, and the reader can find an analogous cycle version of the lemma in Chapter 6. Its elementary proof is based on similar arguments as the arguments that can be used to prove the aforementioned result of Ore, and that form the basis for the well-known Bondy-Chvátal closure for hamiltonicity.

Lemma 2. *Let G be a graph and let P' be an o_{-1}-path in G. Then there is a path P in G such that $V(P') \subset V(P)$.*

Let G be a graph on n vertices. In the following, we denote $\widetilde{E}_{-1}(G) = \{uv : uv \in E(G) \text{ or } d(u) + d(v) \geq n - 1\}$. Let D be an (x_1x_2, y_1y_2)-pair of G. If $x_1x_2 \in \widetilde{E}_{-1}(G)$ or $y_1y_2 \in \widetilde{E}_{-1}(G)$, then using Lemma 2, it is easy to see that G contains a path P with $V(D) \subset V(P)$.

Let G be a graph on n vertices, P be a path of G, $x_1, x, x_2 \in V(P)$ be three distinct vertices appearing in the given order along P, and set $X = V(P[x_1, x_2])$. We say that the pair (x_1, x_2) is *x-good on P*, if for some $j \in \{1, 2\}$, there is a vertex $x' \in X \setminus \{x_j\}$ such that
(1) there is an (x, x_{3-j})-path Q with $V(Q) = X \setminus \{x_j\}$,
(2) there is an $(xx_{3-j}, x'x_j)$-pair D with $V(D) = X$, and
(3) $d(x_j) + d(x') \geq n - 1$.

In this case, we say that Q and D are a path and disjoint path pair *associated with x*, respectively. We present and prove one final useful lemma in this section.

Lemma 3. *Let G be a graph, and P be a path of G. Let $x, y \in V(P)$ and let R be an (x, y)-path in G which is internally-disjoint with P. If there are vertices $x_1, x_2, y_1, y_2 \in V(P) \setminus \{x, y\}$ such that*
(1) *x_1, x, x_2, y_1, y, y_2 appear in this order along P (possibly $x_2 = y_1$),*
(2) *(x_1, x_2) is x-good on P, and*
(3) *(y_1, y_2) is y-good on P,*
then there is a path P' in G such that $V(P) \cup V(R) \subset V(P')$.

Proof. Assume the contrary. Let Q_1 and D_1 be a path and disjoint path pair associated with x, and let Q_2 and D_2 be a path and disjoint path pair associated with y. Let $R' = P[x_2, y_1]$, $R_1 = P[z_1, x_1]$ and $R_2 = P[y_2, z_2]$, where z_1 is the origin and z_2 is the terminus of P.

Using the definition of x-good, we distinguish two main cases and a number of subcases.

Case 1. Q_1 is an (x, x_1)-path, D_1 is an $(xx_1, x'x_2)$-pair, and $d(x_2) + d(x') \geq n - 1$.

Case 1.1. Q_2 is an (y, y_2)-path, D_2 is an $(yy_2, y'y_1)$-pair, and $d(y_1) + d(y') \geq n - 1$.

In this subcase the path $T = R_1 \cup R_2 \cup R \cup R' \cup Q_1 \cup D_2$ is an (z_1z_2, x_2y')-pair which contains all the vertices of $V(P) \cup V(R)$, and $T' = R_1 \cup R_2 \cup R \cup R' \cup Q_2 \cup D_1$ is an $(z_1z_2, x'y_1)$-pair which contains all the vertices of $V(P) \cup V(R)$. Thus by Lemma 2, $d(x_2) + d(y') < n - 1$ and $d(x') + d(y_1) < n - 1$, a contradiction to $d(x_2) + d(x') \geq n - 1$ and $d(y_1) + d(y') \geq n - 1$.

Case 1.2. Q_2 is an (y, y_1)-path, D_2 is an $(yy_1, y'y_2)$-pair, and $d(y_2) + d(y') \geq n - 1$.

Case 1.2.1. The $(xx_1, x'x_2)$-pair D_1 is formed by an (x, x_2)-path and an (x_1, x')-path.

In this subcase, the path $T = R_1 \cup R_2 \cup R \cup R' \cup Q_1 \cup Q_2$ is an (z_1z_2, x_2y_2)-pair which contains all the vertices of $V(P) \cup V(R)$, and the path $T' = R_1 \cup R_2 \cup R \cup R' \cup D_1 \cup D_2$ is an $(z_1z_2, x'y')$-pair which contains all the vertices of $V(P) \cup V(R)$. By Lemma 2, $d(x_2) + d(y_2) < n - 1$ and $d(x') + d(y') < n - 1$, a contradiction.

Case 1.2.2. The $(xx_1, x'x_2)$-pair D_1 is formed by an (x, x')-path and an (x_1, x_2)-path.

Case 1.2.2.1. The $(yy_1, y'y_2)$-pair D_2 is formed by an (y, y_2)-path and an (y_1, y')-path.

This subcase can be proved similarly as Case 1.2.1.

Case 1.2.2.2. The $(yy_1, y'y_2)$-pair D_2 is formed by an (y, y')-path and an (y_1, y_2)-path.

In this subcase, the path $T = R_1 \cup R_2 \cup R \cup R' \cup Q_1 \cup D_2$ is an (z_1z_2, x_2y')-pair which contains all the vertices of $V(P) \cup V(R)$, and the path $T' = R_1 \cup R_2 \cup R \cup R' \cup D_1 \cup Q_1$ is an $(z_1z_2, x'y_2)$-pair which contains all the vertices of $V(P) \cup V(R)$. By Lemma 2, $d(x_2) + d(y') < n - 1$ and $d(x') + d(y_2) < n - 1$, a contradiction.

Case 2. Q_1 is an (x, x_2)-path, D_1 is an $(xx_2, x'x_1)$-pair, and $d(x_1) + d(x') \geq n - 1$.

Case 2.1. Q_2 is an (y, y_2)-path, D_2 is an $(yy_2, y'y_1)$-pair, and $d(y_1) + d(y') \geq n - 1$.

This case can be proved similarly as Case 1.2.

Case 2.2. Q_2 is an (y, y_1)-path, D_2 is an $(yy_1, y'y_2)$-pair, and $d(y_2) + d(y') \geq n - 1$.

In this subcase the path $T = R_1 \cup R_2 \cup R \cup R' \cup D_1 \cup Q_2$ is an (z_1z_2, x_1y')-pair which contains all the vertices of $V(P) \cup V(R)$, and $T' = R_1 \cup R_2 \cup R \cup R' \cup Q_1 \cup D_2$ is an $(z_1z_2, x'y_2)$-pair which contains all the vertices of $V(P) \cup V(R)$. By Lemma 2, $d(x_1) + d(y') < n - 1$ and $d(x') + d(y_2) < n - 1$, a contradiction.

This completes the proof of Lemma 3. □

Let G be a graph with at least one cut vertex and exactly two end blocks, and let P be a path of G. If the two end-vertices of P are inner vertices (not a cut vertex of G) of two distinct end blocks of G, then we call P a *penetrating path* of G. If G is a nonseparable graph, then every path of G is considered to be a penetrating path. Note that a penetrating path of a block-chain G contains all the cut vertices of G, and that a path of a block-chain G is a penetrating path if and only if for every end block of G the path contains at least one inner vertex of the end block.

5.4 Proof of Theorem 5.8

Suppose G is a $\{K_{1,3}, W\}$-o_{-1}-heavy block-chain on n vertices. It suffices to prove that G is traceable. We proceed by contradiction.

Clearly, G contains a penetrating path. Let P be a longest penetrating path of G. We use p to denote the number of vertices of P. Assume that G is not traceable. Then $V(G) \setminus V(P) \neq \emptyset$. Let H be a component of $G - V(P)$. If $N_P(H)$ consists of only one vertex x, then $G[H \cup \{x\}]$ contains an end block of G, contradicting that P is a penetrating path of G. Thus we assume that H has at least two neighbors on P. Let R be a path with two end-vertices on P, all internal vertices in H, and of length at least 2; subject to this, we choose R as short as possible. Suppose without loss of generality, that $P = v_1v_2 \cdots v_p$ and $R = z_0z_1z_2 \cdots z_{r+1}$, where $z_0 = v_s$ and $z_{r+1} = v_t$, $s < t$.

It is easy to see that $N(v_1) \subset V(P)$ and $N(v_p) \subset V(P)$. Thus we have $2 \leq s < t \leq p-1$. We are going to prove ten claims in order to reach a contradiction in all cases.

Claim 1. Let $x \in V(H)$ and $y \in \{v_{s-1}, v_{s+1}, v_{t-1}, v_{t+1}\}$. Then $xy \notin \widetilde{E}_{-1}(G)$.

Proof. Without loss of generality, assume $y = v_{s-1}$ and $xy \in \widetilde{E}_{-1}(G)$. Let Q' be an (x, z_1)-path in H. Then $Q = P[v_1, v_{s-1}]v_{s-1}xQ'z_1v_sP[v_s, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(Q')$. By Lemma 2, there is a path containing all the vertices of $V(P) \cup V(Q')$, which is a longer penetrating path than P, a contradiction. □

Claim 2. $v_{s-1}v_{s+1} \in \widetilde{E}_{-1}(G)$; $v_{t-1}v_{t+1} \in \widetilde{E}_{-1}(G)$.

Proof. If $v_{s-1}v_{s+1} \notin E(G)$, then using Claim 1, the graph induced by $\{v_s, z_1, v_{s-1}, v_{s+1}\}$ is a claw, where $d(z_1)+d(v_{s\pm1}) < n-1$. Since G is a claw-o_{-1}-heavy graph, we have that $d(v_{s-1}) + d(v_{s+1}) \geq n-1$.

The second assertion can be proved similarly. □

Claim 3. $v_{s-1}v_{t-1} \notin \widetilde{E}_{-1}(G)$, $v_{s+1}v_{t+1} \notin \widetilde{E}_{-1}(G)$, $v_sv_{t\pm1} \notin \widetilde{E}_{-1}(G)$, $v_{s\pm1}v_t \notin \widetilde{E}_{-1}(G)$.

Proof. If $v_{s-1}v_{t-1} \in \widetilde{E}_{-1}(G)$, then $Q = P[v_1, v_{s-1}]v_{s-1}v_{t-1}P[v_{t-1}, v_s]v_sRv_t P[v_t, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$. Using Lemma 2, we reach a contradiction.

If $v_sv_{t-1} \in \widetilde{E}_{-1}(G)$, then $Q = P[v_1, v_{s-1}]v_{s-1}v_{s+1}P[v_{s+1}, v_{t-1}]v_{t-1}v_sRv_t P[v_t, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, again a contradiction.

If $v_sv_{t+1} \in \widetilde{E}_{-1}(G)$, then $Q = P[v_1, v_{s-1}]v_{s-1}v_{s+1}P[v_{s+1}, v_t]v_tRv_sv_{t+1} P[v_{t+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, again a contradiction.

The other assertions can be proved similarly. □

Claim 4. Either $v_{s-1}v_{s+1} \in E(G)$ or $v_{t-1}v_{t+1} \in E(G)$.

Proof. Assume the contrary. By Claim 2, we have $d(v_{s-1}) + d(v_{s+1}) \geq n-1$ and $d(v_{t-1}) + d(v_{t+1}) \geq n-1$. By Claim 3, we have $d(v_{s-1}) + d(v_{t-1}) < n-1$ and $d(v_{s+1}) + d(v_{t+1}) < n-1$, a contradiction. □

Now, we distinguish two cases. We treat the case that $r = 1$ and $v_s v_t \in E(G)$ later, but first deal with the case that $r \geq 2$, or $r = 1$ and $v_s v_t \notin E(G)$.

Case 1. $r \geq 2$, or $r = 1$ and $v_s v_t \notin E(G)$.

By Claim 4, without loss of generality, we assume that $v_{s-1}v_{s+1} \in E(G)$. Thus $G[v_{s-1}, v_{s+1}]$ is (v_{s-1}, v_s, v_{s+1})-composed.

Claim 5. $v_s z_2 \notin \widetilde{E}_{-1}(G)$.

Proof. By the choice of the path R, we have $v_s z_2 \notin E(G)$. Now we are going to prove that $d(v_s) + d(z_2) < n - 1$. In order to show this, we first prove a number of subclaims.

Claim 5.1. Every neighbor of v_s is in $V(P) \cup V(H)$; every neighbor of z_2 is in $V(P) \cup V(H)$.

Proof. Assume the contrary. Let $z' \in V(H')$ be a neighbor of v_s, where H' is a component of $G - V(P)$ other than H. Then we have $z'z_1 \notin E(G)$ and $N_{G-P}(z') \cap N_{G-P}(z_1) = \emptyset$.

By Claim 1, we have $v_{s-1}z_1 \notin \widetilde{E}_{-1}(G)$, and similarly, $v_{s-1}z' \notin \widetilde{E}_{-1}(G)$. Thus the graph induced by $\{v_s, v_{s-1}, z_1, z'\}$ is a claw with $d(v_{s-1}) + d(z_1) < n - 1$ and $d(v_{s-1}) + d(z') < n - 1$. Thus we get that $d(z_1) + d(z') \geq n - 1$.

Since $N_{G-P}(z_1) \cap N_{G-P}(z') = \emptyset$, and z, z' are both not adjacent to v_1 and v_p, there exists some i with $2 \leq i \leq p - 2$ such that $z_1 v_i, z' v_{i+1} \in E(G)$. Thus $Q = P[v_1, v_i]v_i z_1 z' v_{i+1} P[v_{i+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup \{z_1, z'\}$. By Lemma 2, there exists a penetrating path containing all the vertices of $V(P) \cup \{z_1, z'\}$, a contradiction.

If $z_2 = v_t$, the second assertion can be proved similarly; and if $z_2 \neq v_t$, the assertion is obvious. □

Let $h = |V(H)|$, and $k = |N_H(v_s)|$. Then we have $d_H(v_s) + d_H(z_2) \leq h + k$. Since $z_1 \in N_H(v_s)$, we have $k \geq 1$. Let $N_H(v_s) = \{y_1, y_2, \ldots, y_k\}$, where $y_1 = z_1$.

Claim 5.2. $y_i y_j \in \widetilde{E}_{-1}(G)$ for all $1 \leq i < j \leq k$.

Proof. If $y_i y_j \notin E(G)$, then by Claim 1, the graph induced by $\{v_s, v_{s-1}, y_i, y_j\}$ is a claw, where $d(y_i) + d(v_{s-1}) < n - 1$ and $d(y_j) + d(v_{s-1}) < n - 1$. Thus we have $d(y_i) + d(y_j) \geq n - 1$. □

Now, let Q' be the o_{-1}-path $Q' = z_2y_1y_2\cdots y_kv_s$. It is clear that $R[z_2, v_t]$ and Q' are internally-disjoint, and Q' contains at least k vertices of H. In the following, we use P' to denote the path $P[v_1, v_{s-1}]v_{s-1}v_{s+1}P[v_{s+1}, v_p]$ if $z_2 \neq v_t$, and to denote the o_{-1}-path $P[v_1, v_{s-1}]v_{s-1}v_{s+1}P[v_{s+1}, v_{t-1}]v_{t-1}v_{t+1}P[v_{t+1}, v_p]$ if $z_2 = v_t$.

Claim 5.3. If $v_sv_i \in E(G)$ for some i with $2 \leq i \leq p-1$, then $z_2v_{i-1}, z_2v_{i+1} \notin E(G)$.

Proof. If $v_sv_i \in E(G)$ for some i with $2 \leq i \leq p-1$ and $z_2v_{i-1} \in E(G)$, then $Q = P'[v_1, v_{i-1}]v_{i-1}z_2Q'v_sv_iP'[v_i, v_p]$ is an o_{-1}-path containing all vertices of $V(P) \cup V(Q')$, another contradiction (using Lemma 2).

Similarly, we can prove the assertion for z_2v_{i+1}. □

By Claim 3, we have $v_sv_{t-1} \notin E(G)$. Let v_ℓ be the last vertex in $P[v_{s+1}, v_{t-1}]$ such that $v_sv_\ell \in E(G)$.

Claim 5.4. $t - \ell \geq k + 1$, and for every vertex $v_i \in [v_{\ell+1}, v_{\ell+k}]$, $z_2v_i \notin E(G)$.

Proof. If $t - \ell \leq k$, then $Q = P[v_1, v_{s-1}]v_{s-1}v_{s+1}P[v_{s+1}, v_\ell]v_\ell v_sQ'z_2R[z_2, v_t]v_t P[v_t, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(Q') \cup V(R) \setminus [v_{\ell+1}, v_{t-1}]$, which yields a longer penetrating path than P, using Lemma 2, a contradiction.

If $z_2v_i \notin E(G)$ for some $v_i \in [v_{\ell+1}, v_{\ell+k}]$, then $Q = P'[v_1, v_\ell]v_\ell v_sQ'z_2v_i P'[v_i, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(Q') \setminus [v_{\ell+1}, v_{i-1}]$, which again yields a longer penetrating path than P, a contradiction. □

Claim 5.5. $d_{[v_{s+1},v_{t-1}]}(v_s) + d_{[v_{s+1},v_{t-1}]}(z_2) \leq t - s - k - 1$; $d_{[v_1,v_{s-1}]}(v_s) + d_{[v_1,v_{s-1}]}(z_2) \leq s - 2$; $d_{[v_{t+1},v_p]}(v_s) + d_{[v_{t+1},v_p]}(z_2) \leq p - t - 1$.

Proof. Note that $v_sv_{t-1} \notin E(G)$ and $z_2v_{s+1} \notin E(G)$. If v_s has d neighbors in $[v_{s+1}, v_{t-2}]$, then by Claims 5.3 and 5.4, z_2 has at most $t - s - 2 - d - k + 1$ neighbors in $[v_{s+2}, v_{t-1}]$.

Note that $z_2v_{s-1} \notin E(G)$ and $v_sv_1 \notin E(G)$. If v_s has d neighbors in $[v_2, v_{s-1}]$, then by Claim 5.3, z_2 has at most $s - 2 - d$ neighbors in $[v_1, v_{s-2}]$.

Similarly, note that $v_sv_{t+1} \notin E(G)$ and $z_2v_p \notin E(G)$. If z_2 has d neighbors in $[v_{t+1}, v_{p-1}]$, then by Claim 5.3, v_s has at most $p - t - 1 - d$ neighbors in $[v_{t+2}, v_p]$. □

Now we can complete the proof of Claim 5. Note that v_s and z_2 are possibly adjacent to v_t, but they cannot be adjacent to v_s. By Claim 5.3, we have $d_P(v_s) + d_P(z_2) \leq p - k - 2$. Recall that $d_H(v_s) + d_H(z_2) \leq h + k$. By Claim 5.1, we have that $d(v_s) + d(z_2) \leq p + h - 2 < n - 1$. □

Recall that $G[v_{s-1}, v_{s+1}]$ is (v_{s-1}, v_s, v_{s+1})-composed. Now we prove the following claims.

Claim 6. If $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed with canonical path $P[v_{s-k}, v_{s+\ell}]$, then $s - k \geq 2$ and $s + \ell \leq t - 3$.

Proof. Let $D_1, D_2, \ldots, D_r$ be a canonical sequence of $G[v_{s-k}, v_{s+\ell}]$ corresponding to the canonical path $P[v_{s-k}, v_{s+\ell}]$. If $s-k = 1$, then by Lemma 1, there is a Hamilton $(v_s, v_{s+\ell})$-path Q' of $G[v_{s-k}, v_{s+\ell}]$. Thus $Q = z_1 v_s Q' v_{s+\ell} P[v_{s+\ell}, v_p]$ is a path containing all the vertices of $V(P) \cup \{z_1\}$, a contradiction.

If $s + \ell \geq t - 2$, then consider the graph D_i, where i is the smallest integer such that $v_{t-2} \in V(D_i)$. Let $V(D_i) = [v_{s-k'}, v_{t-2}]$. By Lemma 1, there exists a Hamilton $(v_{s-k'}, v_s)$-path Q' of $G[v_{s-k'}, v_{t-2}]$. Thus $Q = P[v_1, v_{s-k'}] v_{s-k'} Q' v_s R v_t v_{t-1} v_{t+1} P[v_{t+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, yielding another contradiction. □

Claim 7. If $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed with canonical path $P[v_{s-k}, v_{s+\ell}]$, where $s - k \geq 2$ and $s + \ell \leq t - 3$, and any two nonadjacent vertices in $[v_{s-k-1}, v_{s+\ell+1}]$ have degree sum less than $n - 1$, then one of the following is true:

(1) $G[v_{s-k-1}, v_{s+\ell}]$ is a 1-extension of $G[v_{s-k}, v_{s+\ell}]$ at v_{s-k} to v_{s-k-1};
(2) $G[v_{s-k}, v_{s+\ell+1}]$ is a 1-extension of $G[v_{s-k}, v_{s+\ell}]$ at $v_{s+\ell}$ to $v_{s+\ell+1}$; or
(3) $G[v_{s-k-1}, v_{s+\ell+1}]$ is a 2-extension of $G[v_{s-k}, v_{s+\ell}]$ at $(v_{s-k}, v_{s+\ell})$ to $(v_{s-k-1}, v_{s+\ell+1})$.

Thus in all cases we obtain a composed graph larger than $G[v_{s-k}, v_{s+\ell}]$.

Proof. Assume the contrary. This implies that v_{s-k-1} has only one neighbor v_{s-k}, and $v_{s+\ell+1}$ has only one neighbor $v_{s+\ell}$ in $[v_{s-k-1}, v_{s+\ell+1}]$. We prove a number of subclaims in order to reach contradictions in all cases.

Claim 7.1. Let $i \in [s-k-1, s+\ell+1] \setminus \{s\}$ and $j = 1, 2$. Then $v_i z_j \notin \widetilde{E}_{-1}(G)$.

Proof. Without loss of generality, we assume that $i < s$. If $i = s - 1$, the assertion is true by Claims 1 and 3. So we assume that $i \in [s - k - 1, s - 2]$ and $i+1 \in [s-k, s-1]$. By the definition of composed subgraphs, there exists an $i' \in [s+1, s+\ell]$ such that $G[v_i, v_{i'}]$ is $(v_i, v_s, v_{i'})$-composed. By Lemma 1, there exists a Hamilton $(v_s, v_{i'})$-path Q' of $G[v_i, v_{i'}]$.

If $z_j \neq v_t$, then $Q = P[v_1, v_i]v_iz_jR[z_j, v_s]v_sQ'v_{i'}P[v_{i'}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup \{z_j\}$, yielding another contradiction.

If $z_j = v_t$, then $Q = P[v_1, v_i]v_iv_tRv_sQ'v_{i'}P[v_{i'}, v_{t-1}]v_{t-1}v_{t+1}P[v_{t+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, yielding another contradiction. □

Let $G' = G[[v_{s-k-1}, v_{s+\ell}] \cup \{z_1, z_2\}]$ and $G'' = G[[v_{s-k-1}, v_{s+\ell+1}] \cup \{z_1, z_2\}]$.

Claim 7.2. G'' and G' are $\{K_{1,3}, W\}$-free.

Proof. By Claims 5 and 7.1, and the condition that any two nonadjacent vertices in $[v_{s-k-1}, v_{s+\ell+1}]$ have degree sum less than $n-1$, we have that any two nonadjacent vertices in G'' have degree sum less than $n-1$. Since G (and hence G'') is $\{K_{1,3}, W\}$-o_{-1}-heavy, we have that G'' is $\{K_{1,3}, W\}$-free. The second assertion follows easily. □

Claim 7.3. $N_{G'}(v_s) \setminus \{z_1\}$ is a clique.

Proof. If there are two vertices $x, x' \in N_{G'}(v_s) \setminus \{z_1\}$ such that $xx' \notin E(G')$, then the graph induced by $\{v_s, z_1, x, x'\}$ is a claw, a contradiction. □

Now, we define $N_i = \{x \in V(G') : d_{G'}(x, v_{s-k-1}) = i\}$. Then $N_0 = \{v_{s-k-1}\}$, $N_1 = \{v_{s-k}\}$ and $N_2 = N_{G'}(v_{s-k}) \setminus \{v_{s-k-1}\}$.

By the definition of composed subgraphs, we have $|N_2| \geq 2$. If there are two vertices $x, x' \in N_2$ such that $xx' \notin E(G')$, then the graph induced by $\{v_{s-k}, v_{s-k-1}, x, x'\}$ is a claw, a contradiction. Thus N_2 is a clique.

We assume $v_s \in N_j$, where $j \geq 2$. Then $z_1 \in N_{j+1}$ and $z_2 \in N_{j+2}$.

If $|N_i| = 1$ for some $i \in [2, j-1]$, let $N_i = \{x\}$. Then x is a cut vertex of the graph $G[v_{s-k}, v_{s+\ell}]$. By the definition of composed subgraphs, $G[v_{s-k}, v_{s+\ell}]$ is 2-connected. This implies $|N_i| \geq 2$ for every $i \in [2, j-1]$.

Claim 7.4. For $i \in [1, j]$, N_i is a clique.

Proof. We prove this claim by induction on i. For $i = 1, 2$, the claim is true by the above analysis. So we assume that $3 \leq i \leq j$, and we have that $N_{i-3}, N_{i-2}, N_{i-1}, N_{i+1}$ and N_{i+2} are nonempty, and that $|N_{i-1}| \geq 2$.

Let x be a vertex in N_i that has a neighbor y in N_{i+1}. We claim that for every $x' \in N_i$, $xx' \in E(G)$. Suppose that $xx' \notin E(G)$. If x and x' have a common neighbor in N_{i-1}, denote it by w; then let v be a neighbor of w in N_{i-2}, and the graph induced by $\{w, v, x, x'\}$ is a claw, a contradiction. Thus x and x' have no common neighbors in N_{i-1}. Now, let w be a neighbor of x in N_{i-1}, and let w' be a neighbor of x' in N_{i-1}. Then $xw', x'w \notin E(G)$. Let v be a neighbor of w in N_{i-2}, and let u be a neighbor of v in N_{i-3}. If $w'v \notin E(G)$, then the graph induced by $\{w, v, w', x\}$ is a claw, a contradiction. Thus $w'v \in E(G)$, and then the graph induced by $\{w', v, u, w, x, y\}$ is a W, a contradiction. Thus as we claimed, x is adjacent to every other vertex in N_i.

Now, we claim that for every two distinct vertices x' and x'' in N_i other than x, $x'x'' \in E(G)$. Supposed that $x'x'' \notin E(G)$. If $x'y \in E(G)$, then similarly as before, we can prove that x' is adjacent to any other vertices in N_i; then $x'x'' \in E(G)$. Thus we assume that $x'y \notin E(G)$, and similarly, $x''y \notin E(G)$. Then the graph induced by $\{x, x', x'', y\}$ is a claw, a contradiction. This implies that N_i is a clique. □

If there exists some vertex $x \in N_j$ other than v_s, then $v_s x \in E(G)$ by Claim 7.4. Let w be a neighbor of v_s in N_{j-1}, and let v be a neighbor of w in N_{j-2}. Then $wx \in E(G)$ by Claim 7.3. Thus the graph induced by $\{x, w, v, v_s, z_1, z_2\}$ is a W, a contradiction. So we assume N_j consists of only one vertex v_s.

If there exists some vertex $x \in N_{j+1}$ other than z_1, then v_s is a cut vertex of the graph $G[v_{s-k}, v_{s+\ell}]$, a contradiction. So we assume that all vertices in $[u_{-k}, u_\ell]$ are in $\bigcup_{i=1}^{j} N_i$.

Let $v_{s+\ell} \in N_i$, where $i \in [2, j-1]$. If $v_{s+\ell}$ has a neighbor in N_{i+1}, then let y be a neighbor of $v_{s+\ell}$ in N_{i+1}, and let w be a neighbor of $v_{s+\ell}$ in N_{i-1}. Then the graph induced by $\{v_{s+\ell}, w, y, v_{s+\ell+1}\}$ is a claw, a contradiction. So $v_{s+\ell}$ has no neighbors in N_{i+1}.

Let z be a vertex in N_{i+2}, let y be a neighbor of z in N_{i+1}, let x be a neighbor of y in N_i, and let w be a neighbor of x in N_{i-1}. Thus $x \neq v_{s+\ell}$. If $wv_{s+\ell} \notin E(G)$, then the graph induced by $\{x, w, v_{s+\ell}, y\}$ is a claw, a contradiction. So $wv_{s+\ell} \in E(G)$ and the graph induced by $\{w, v_{s+\ell}, v_{s+\ell+1}, x, y, z\}$ is a W, a contradiction. This final contradiction completes the proof of Claim 7. □

Using Claim 7, we can consider a largest composed subgraph, in the following sense. We choose k, ℓ such that:

(1) $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed with canonical path $P[v_{s-k}, v_{s+\ell}]$;

(2) any two nonadjacent vertices in $[v_{s-k}, v_{s+\ell}]$ have degree sum less than $n-1$; and

(3) $k+\ell$ is as large as possible.

Claim 8. $(v_{s-k-1}, v_{s+\ell})$ or $(v_{s-k}, v_{s+\ell+1})$ or $(v_{s-k-1}, v_{s+\ell+1})$ is v_s-good on P.

Proof. By Claim 7, there exists a vertex $v_i \in [v_{s-k+1}, v_{s+\ell}]$ such that $d(v_{s-k-1})+d(v_i) \geq n-1$, or there exists a vertex $v_i \in [v_{s-k}, v_{s+\ell-1}]$ such that $d(v_{s+\ell+1})+d(v_i) \geq n-1$, or $d(v_{s-k-1})+d(v_{s+\ell+1}) \geq n-1$.

Suppose first there exists a vertex $v_i \in [v_{s-k+1}, v_{s+\ell}]$ with $d(v_{s-k-1})+d(v_i) \geq n-1$. Since $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed, by Lemma 1, there exists a $(v_s, v_{s+\ell})$-path Q such that $V(Q)=[v_{s-k}, v_{s+\ell}]$, and there exists a $(v_s v_{s+\ell}, v_i v_{s-k})$-pair D' such that $V(D')=[v_{s-k}, v_{s+\ell}]$, and $D=D' \cup \{v_{s-k} v_{s-k-1}\}$ is a $(v_s v_{s+\ell}, v_i v_{s-k-1})$-pair such that $V(D)=[v_{s-k-1}, v_{s+\ell}]$. Thus $(v_{s-k-1}, v_{s+\ell})$ is v_s-good on P.

If there exists a vertex $v_i \in [v_{s-k}, v_{s+\ell-1}]$ with $d(v_{s+\ell+1})+d(v_i) \geq n-1$, we can prove the result similarly.

Now suppose that $d(v_{s-k-1})+d(v_{s+\ell+1}) \geq n-1$. Since $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed, by Lemma 1, there exists a $(v_s, v_{s+\ell})$-path Q' such that $V(Q')=[v_{s-k}, v_{s+\ell}]$, and there exists a (v_s, v_{s-k})-path Q'' such that $V(Q'')=[v_{s-k}, v_{s+\ell}]$. Then $Q=Q' v_{s+\ell} v_{s+\ell+1}$ is a $(v_s, v_{s+\ell+1})$-path such that $V(Q)=[v_{s-k}, v_{s+\ell+1}]$, and $D=Q'' v_{s-k} v_{s-k-1} \cup v_{s+\ell+1}$ is a $(v_s v_{s-k-1}, v_{s+\ell+1} v_{s-k-1})$-pair such that $V(D)=[v_{s+\ell+1}, v_{s-k-1}]$. Thus $(v_{s+\ell+1}, v_{s-k-1})$ is v_s-good on P. □

Claim 9. There exist some k' and ℓ' such that $(v_{t-k'}, v_{t+\ell'})$ is v_t-good on P, where $s+\ell+1 \leq t-k'$ and $t+\ell' \leq p$.

Proof. By Claim 6, we have $s+\ell \leq t-3$.

If $v_{t-1} v_{t+1} \notin E(G)$, then by Claim 2, $d(v_{t-1})+d(v_{t+1}) \geq n-1$. Then $Q=v_t v_{t-1}$ is a (v_t, v_{t-1})-path and $D=v_t v_{t+1} \cup v_{t-1}$ is a $(v_t v_{t-1}, v_{t-1} v_{t+1})$-pair. Thus we have that (v_{t-1}, v_{t+1}) is v_t-good on P.

Now we assume that $v_{t-1} v_{t+1} \in E(G)$, and then $G[v_{t-1}, v_{t+1}]$ is (v_{t-1}, v_t, v_{t+1})-composed.

Claim 9.1. If $G[v_{t-k'}, v_{t+\ell'}]$ is $(v_{t-k'}, v_t, v_{t+\ell'})$-composed with canonical path $P[v_{t-k'}, v_{t+\ell'}]$, then $t-k' \geq s+\ell+2$ and $t+\ell' \leq p-1$.

Proof. Let $D_1, D_2, \ldots, D_r$ be a canonical sequence of $G[v_{-k'}, v_{\ell'}]$ corresponding to the canonical path $P[v_{t-k'}, v_{t+\ell'}]$. Similarly as in the proof of Claim 6, we have that $t + \ell' \leq p - 1$. Suppose now that $t - k' \leq s + \ell + 1$. Consider the graph D_i, where i is the smallest integer such that $v_{s+\ell+1} \in V(D_i)$. Let $V(D_i) = [v_{s+\ell+1}, v_{t+\ell''}]$. By Lemma 1, there exists a Hamilton (v_s, v_{s-k})-path Q' of $G[v_{s-k}, v_{s+\ell}]$ and there exists a Hamilton path Q'' of $G[v_{s+\ell+1}, v_{t+\ell''}]$. Thus $Q = P[v_1, v_{s-k}]v_{s-k}Q'v_sRv_tQ''v_{t+\ell''}P[v_{t+\ell''}, v_p]$ is a path containing all the vertices of $V(P) \cup V(R)$, a contradiction. □

Similar to Claim 7, we have another claim that provides a tool for considering a largest composed subgraph.

Claim 9.2. If $G[v_{t-k'}, v_{t+\ell'}]$ is $(v_{t-k'}, v_t, v_{t+\ell'})$-composed with canonical path $P[v_{t-k'}, v_{t+\ell'}]$, where $t-k' \geq s+\ell+2$ and $t+\ell \leq p-1$, and any two nonadjacent vertices in $[v_{t-k'-1}, v_{t+\ell'+1}]$ have degree sum less than $n-1$, then one of the following is true:

(1) $G[v_{t-k'-1}, v_{t+\ell'}]$ is a 1-extension of $G[v_{t-k'}, v_{t+\ell'}]$ at $v_{t-k'}$ to $v_{t-k'-1}$;
(2) $G[v_{t-k'}, v_{t+\ell'+1}]$ is a 1-extension of $G[v_{t-k'}, v_{t+\ell'}]$ at $v_{t+\ell'}$ to $v_{t+\ell'+1}$; or
(3) $G[v_{t-k'-1}, v_{t+\ell'+1}]$ is a 2-extension of $G[v_{t-k'}, v_{t+\ell'}]$ at $(v_{t-k'}, v_{t+\ell'})$ to $(v_{t-k'-1}, v_{t+\ell'+1})$.

Hence in all cases we obtain a composed graph larger than $G[v_{t-k'}, v_{t+\ell'}]$.

Now we choose k', ℓ' such that:

(1) $G[v_{t-k'}, v_{t+\ell'}]$ is $(v_{t-k'}, v_t, v_{t+\ell'})$-composed with canonical path $P[v_{t-k'}, v_{t+\ell'}]$;
(2) any two nonadjacent vertices in $[v_{t-k'}, v_{t+\ell'}]$ have degree sum less than $n-1$; and
(3) $k' + \ell'$ is as large as possible.

Similar to Claim 8, we have that $(v_{t-k'-1}, v_{t+\ell'})$ or $(v_{t-k'}, v_{t+\ell'+1})$ or $(v_{t-k'-1}, v_{t+\ell'+1})$ is v_t-good on P. □

Using Claims 8 and 9, by Lemma 3, we get that there exists a path containing all the vertices of $V(P) \cup V(R)$, a contradiction. This completes the proof for Case 1.

Case 2. $r = 1$ and $v_sv_t \in E(G)$.

Recall that $v_sv_{s+1} \in E(G)$ and $v_sv_{t-1} \notin E(G)$. Let v_{s+k} be the first vertex in $[v_{s+1}, v_{t-1}]$ such that $v_sv_{s+k} \notin E(G)$. Then $s+2 \leq s+k \leq t-1$.

Claim 10. Let $v_i \in [v_{s+1}, v_{s+k}]$ and $x \in \{z_1, v_t, v_{t+1}\}$. Then $v_i x \notin \widetilde{E}_{-1}(G)$.

Proof. By Claims 1 and 3, we have that $v_{s+1}z_1, v_{s+1}v_t, v_{s+1}v_{t+1} \notin \widetilde{E}_{-1}(G)$. Thus we assume that $v_i \in [v_{s+2}, v_{s+k}]$. Then $v_s v_i \in E(G)$. If $v_i z_1 \in \widetilde{E}(G)$, then $Q = P[v_1, v_{s-1}]v_{s-1}\ v_{s+1}P[v_{s+1}, v_{i-1}]v_{i-1}v_s z_1 v_i P[v_i, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, yielding a contradiction using Lemma 2. If $v_i v_t \in \widetilde{E}(G)$, then $Q = P[v_1, v_{s-1}]v_{s-1}v_{s+1}\ P[v_{s+1}, v_{i-1}]v_{i-1}v_s z_1 v_t v_i P[v_i, v_{t-1}]v_{t-1}v_{t+1}P[v_{t+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, yielding another contradiction. If $v_i v_{t+1} \in \widetilde{E}(G)$, then $Q = P[v_1, v_{s-1}]\ v_{s-1}v_{s+1}P[v_{s+1}, v_{i-1}]v_{i-1}v_s z_1 v_t P[v_t, v_i]v_i v_{t+1}P[v_{t+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, yielding another contradiction. □

Using Claims 1, 3 and 10, the subgraph induced by $\{z_1, v_t, v_{t+1}, v_s, v_{s+k-1}, v_{s+k}\}$ is a W that is not o_{-1}-heavy, our final contradiction, completing the proof of Theorem 5.9.

5.5 Proof of Theorem 5.9

The proof is modelled along the same lines as the proof of Theorem 5.9. Suppose G is a $\{K_{1,3}, N\}$-o_{-1}-heavy block-chain on n vertices. It suffices to prove that G is traceable. We proceed by contradiction.

Clearly, G contains a penetrating path. As in the previous section, we choose a longest penetrating path $P = v_1 v_2 \cdots v_p$, a component H of $G - V(P)$, and a path $R = z_0 z_1 z_2 \cdots z_{r+1}$, where $z_0 = v_s$ and $z_{r+1} = v_t$, $s < t$ with two end-vertices on P and all internal vertices in H, and of length at least 2, but as short as possible subject to this.

Similarly as in Section 5.4, we get the following claims. We omit the details.

Claim 1. Let $x \in V(H)$ and $y \in \{v_{s-1}, v_{s+1}, v_{t-1}, v_{t+1}\}$. Then $xy \notin \widetilde{E}_{-1}(G)$.

Claim 2. $v_{s-1}v_{s+1} \in \widetilde{E}_{-1}(G)$; $v_{t-1}v_{t+1} \in \widetilde{E}_{-1}(G)$.

Claim 3. $v_{s-1}v_{t-1} \notin \widetilde{E}_{-1}(G)$, $v_{s+1}v_{t+1} \notin \widetilde{E}_{-1}(G)$, $v_s v_{t\pm1} \notin \widetilde{E}_{-1}(G)$, $v_{s\pm1}v_t \notin \widetilde{E}_{-1}(G)$.

Claim 4. Either $v_{s-1}v_{s+1} \in E(G)$ or $v_{t-1}v_{t+1} \in E(G)$.

By Claim 4, without loss of generality, we assume that $v_{s-1}v_{s+1} \in E(G)$. Thus $G[v_{s-1}, v_{s+1}]$ is (v_{s-1}, v_s, v_{s+1})-composed.

Claim 5. If $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed with canonical path $P[v_{s-k}, v_{s+\ell}]$, then $s-k \geq 2$ and $s+\ell \leq t-3$.

The proof of Claim 5 is similar to that of Claim 6 in Section 5.4.

Now we prove the following claim.

Claim 6. If $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed with canonical path $P[v_{s-k}, v_{s+\ell}]$, where $s-k \geq 2$ and $s+\ell \leq t-3$, and any two nonadjacent vertices in $[v_{s-k-1}, v_{s+\ell+1}]$ have degree sum less than $n-1$, then one of the following is true:

(1) $G[v_{s-k-1}, v_{s+\ell}]$ is a 1-extension of $G[v_{s-k}, v_{s+\ell}]$ at v_{s-k} to v_{s-k-1};

(2) $G[v_{s-k}, v_{s+\ell+1}]$ is a 1-extension of $G[v_{s-k}, v_{s+\ell}]$ at $v_{s+\ell}$ to $v_{s+\ell+1}$; or

(3) $G[v_{s-k-1}, v_{s+\ell+1}]$ is a 2-extension of $G[v_{s-k}, v_{s+\ell}]$ at $(v_{s-k}, v_{s+\ell})$ to $(v_{s-k-1}, v_{s+\ell+1})$.

Thus in all cases we obtain a composed graph larger than $G[v_{s-k}, v_{s+\ell}]$.

Proof. Assume the contrary. This implies that v_{s-k-1} has only one neighbor v_{s-k}, and $v_{s+\ell+1}$ has only one neighbor $v_{s+\ell}$, in $[v_{s-k-1}, v_{s+\ell+1}]$. We need a number of subclaims.

Claim 6.1. For $i \in [s-k-1, s+\ell+1] \setminus \{s\}$, $v_i z_1 \notin \widetilde{E}_{-1}(G)$.

This claim can be proved in a similar way as Claim 7.1 in Section 5.4. We omit the details.

Let $G' = G[[v_{s-k-1}, v_{s+\ell}] \cup \{z_1\}]$ and $G'' = G[[v_{s-k-1}, v_{s+\ell+1}] \cup \{z_1\}]$.

Similar to Claims 7.2 and 7.3 in Section 5.4, we obtain the following statements.

Claim 6.2. G'' and G' are $\{K_{1,3}, N\}$-free.

Claim 6.3. $N_{G'}(v_s) \setminus \{z_1\}$ is a clique.

Now, we define $N_i = \{x \in V(G') : d_{G'}(x, v_{s-k-1}) = i\}$. Then $N_0 = \{v_{s-k-1}\}$, $N_1 = \{v_{s-k}\}$ and $N_2 = N_{G'}(v_{s-k}) \setminus \{v_{s-k-1}\}$.

By the definition of composed graphs, we have $|N_2| \geq 2$. If there are two vertices $x, x' \in N_2$ such that $xx' \notin E(G')$, then the graph induced by $\{v_{s-k}, v_{s-k-1}, x, x'\}$ is a claw, a contradiction. Thus N_2 is a clique.

We assume $v_s \in N_j$, where $j \geq 2$. Then $z_1 \in N_{j+1}$.

If $|N_i| = 1$ for some $i \in [2, j-1]$, then let $N_i = \{x\}$; then x is a cut vertex of the graph $G[v_{s-k}, v_{s+\ell}]$. By the definition of composed graphs, $G[v_{s-k}, v_{s+\ell}]$ is 2-connected. This implies $|N_i| \geq 2$ for every $i \in [2, j-1]$.

Claim 6.4. For $i \in [1, j]$, N_i is a clique.

Proof. We prove this claim by induction on i. For $i = 1, 2$, the claim is true by the above analysis. So we assume that $3 \leq i \leq j$, and we have that $N_{i-3}, N_{i-2}, N_{i-1}$ and N_{i+1} are nonempty, and that $|N_{i-1}| \geq 2$.

Let x and x' be two distinct vertices in N_i. We claim that $xx' \in E(G)$. Suppose that $xx' \notin E(G)$. If x and x' have a common neighbor in N_{i-1}, denote it by w; then let v be a neighbor of w in N_{i-2}, and the graph induced by $\{w, v, x, x'\}$ is a claw, a contradiction. Thus x and x' have no common neighbors in N_{i-1}. Now, let w be a neighbor of x in N_{i-1}, and let w' be a neighbor of x' in N_{i-1}. Then $xw', x'w \notin E(G)$. Let v be a neighbor of w in N_{i-2}, and let u be a neighbor of v in N_{i-3}. If $w'v \notin E(G)$, then the graph induced by $\{w, v, w', x\}$ is a claw, a contradiction. Thus $w'v \in E(G)$, and then the graph induced by $\{v, u, w, x, w', x'\}$ is an N, a contradiction. This implies that N_i is a clique. □

If there exists some vertex $y \in N_{j+1}$ other than z_1, then we have $yv_s \notin E(G)$ by Claim 6.3. Let x be a neighbor of y in N_j, let w be a neighbor of v_s in N_{j-1}, and let v be a neighbor of w in N_{j-2}. Then $xv_s \in E(G)$ by Claim 6.4, and $xw \in E(G)$ by Claim 6.3. Thus the graph induced by $\{w, v, x, y, v_s, z_1\}$ is an N, a contradiction. So we assume that all vertices in $[v_{s-k}, v_{s+\ell}]$ are in $\bigcup_{i=1}^{j} N_i$.

If $v_{s+\ell} \in N_j$, then let w be a neighbor of v_s in N_{j-1}, and let v be a neighbor of w in N_{j-2}. Then the graph induced by $\{w, v, v_{s+\ell}, v_{s+\ell+1}, v_s, z_1\}$ is an N, a contradiction. Thus $v_{s+\ell} \notin N_j$ and thus $j \geq 3$.

Let $v_{s+\ell} \in N_i$, where $i \in [2, j-1]$. If $v_{s+\ell}$ has a neighbor in N_{i+1}, then let y be a neighbor of $v_{s+\ell}$ in N_{i+1}, and let w be a neighbor of $v_{s+\ell}$ in N_{i-1}. Then the graph induced by $\{v_{s+\ell}, w, y, v_{s+\ell+1}\}$ is a claw, a contradiction. Thus $v_{s+\ell}$ has no neighbors in N_{i+1}.

Let y be a vertex in N_{i+1}, and let x be a neighbor of y in N_i. Then $x \neq v_{s+\ell}$. Let w be a neighbor of x in N_{i-1}, and let v be a neighbor of w in N_{i-2}. If $wv_{s+\ell} \notin E(G)$, then the graph induced by $\{x, w, v_{s+\ell}, y\}$ is a claw, a contradiction. So $wv_{s+\ell} \in E(G)$ and the graph induced by $\{w, v, v_{s+\ell}, v_{s+\ell+1}, x, y\}$ is an N, a contradiction.

This completes the proof of Claim 6. □

Using Claim 6, we consider a largest composed subgraph, in the following sense. We choose k, ℓ such that:

(1) $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed with canonical path $P[v_{s-k}, v_{s+\ell}]$;
(2) any two nonadjacent vertices in $[v_{s-k}, v_{s+\ell}]$ have degree sum less than $n-1$; and
(3) $k+\ell$ is as large as possible.

Similar to Claims 8 and 9 in Section 5.4, we obtain the following claims. We omit the details.

Claim 7. $(v_{s-k-1}, v_{s+\ell})$ or $(v_{s-k}, v_{s+\ell+1})$ or $(v_{s-k-1}, v_{s+\ell+1})$ is v_s-good on P.

Claim 8. There exist some k' and ℓ' such that $(v_{t-k'}, v_{t+\ell'})$ is v_t-good on P, where $s+\ell+1 \leq t-k'$ and $t+\ell' \leq p$.

Using Claims 7 and 8, Lemma 3 implies that there exists a path containing all the vertices of $V(P) \cup V(R)$, our final contradiction.

Chapter 6

Heavy pairs for hamiltonicity

6.1 Introduction

In this chapter, we consider the hamiltonicity of 2-connected graphs. The following characterization of pairs of forbidden subgraphs for the existence of Hamilton cycles in graphs is well-known. We refer to Figure 6.1 for an illustration of the graphs appearing in the next result.

Theorem 6.1 (Bedrossian [3]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, P_6, C_3, Z_1, Z_2, B, N or W.*

Our aim in this chapter is to consider the corresponding heavy subgraph conditions for a graph to be hamiltonian. First, we notice that every 2-connected P_3-heavy graph contains a Hamilton cycle. This can be easily deduced from the following result due to Fan.

Theorem 6.2 (Fan [22]). *Let G be a 2-connected graph. If* $\max\{d(u), d(v)\} \geq n/2$ *for every pair of vertices at distance 2 in G, then G is hamiltonian.*

It is not difficult to see that P_3 is the only connected graph S such that every 2-connected S-heavy graph is hamiltonian. So we have the following natural open problem.

Problem 6.1. Which two connected graphs R and S other than P_3 imply that every 2-connected $\{R, S\}$-heavy graph is hamiltonian?

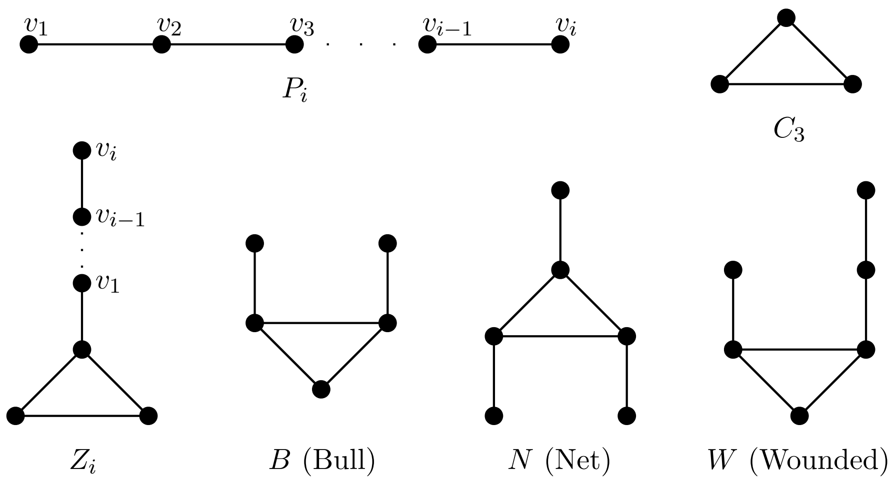


Figure 6.1: Graphs P_i, C_3, Z_i, B, N and W

By Theorem 6.1, we get that (up to symmetry) $R = K_{1,3}$ and S must be one of the graphs P_4, P_5, P_6, C_3, Z_1, Z_2, B, N or W.

As will be shown in the sequel, we get the following results.

Theorem 6.3. *If G is a 2-connected $\{K_{1,3}, W\}$-heavy graph, then G is hamiltonian.*

Theorem 6.4. *If G is a 2-connected $\{K_{1,3}, N\}$-heavy graph, then G is hamiltonian.*

The graph family illustrated in Figure 6.2 consists of members that are 2-connected, $\{K_{1,3}, P_6\}$-heavy and not hamiltonian. This is easy to check.

We can also construct 2-connected claw-free and P_6-heavy graphs that are not hamiltonian. This can be shown as follows: Let G be a graph from Figure 6.2, where $r \geq 15$ is an integer divisible by 3. Let V_1, V_2, V_3 be a balanced partition of K_r, and let G' be the graph obtained from G by deleting all the edges in $\bigcup_{i=1}^{3}\{x_i v : v \in V_i\}$. Then G' is a 2-connected claw-free and P_6-heavy graph that is not hamiltonian.

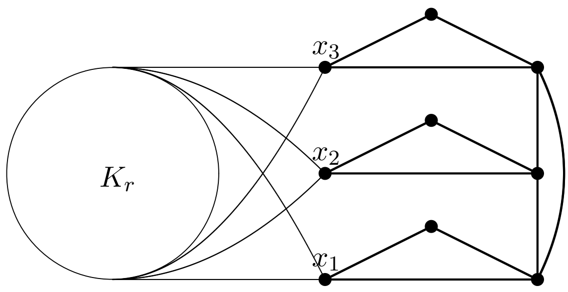


Figure 6.2: A 2-connected $\{K_{1,3}, P_6\}$-heavy non-Hamiltonian graph ($r \geq 5$)

Note that W contains induced copies of P_4, P_5, C_3, Z_1, Z_2 and B. So we have obtain the following result.

Theorem 6.5. *Let R and S be connected graphs with $R, S \neq P_3$, and let G be a 2-connected graph. Then G being $\{R, S\}$-heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, C_3, Z_1, Z_2, B, N or W.*

Thus, Theorem 6.5 gives a complete answer to Problem 6.1.

For claw-heavy graphs, Chen et al. obtained the following result. Here we refer to Figure 6.3 for an illustration of the graphs D and H.

Theorem 6.6 (Chen, Zhang and Qiao [18])**.** *Let G be a 2-connected graph. If G is claw-heavy and, moreover $\{P_7, D\}$-free or $\{P_7, H\}$-free, then G is hamiltonian.*

It is clear that every P_6-free graph is also $\{P_7, D\}$-free. Thus every 2-connected claw-heavy and P_6-free graph is hamiltonian. Together with Theorems 6.3 and 6.4, we obtain the following characterization.

Theorem 6.7. *Let S be a connected graph with $S \neq P_3$ and let G be a 2-connected claw-heavy graph. Then G being S-free implies G is hamiltonian if and only if $S = P_4$, P_5, P_6, C_3, Z_1, Z_2, B, N or W.*

The 'only-if' part of the above theorem follows immediately from Theorem 6.1.

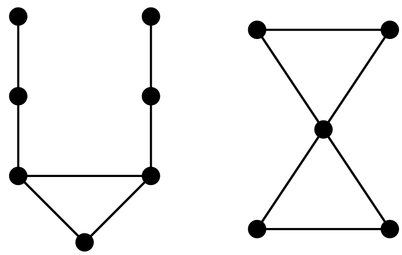


Figure 6.3: Graphs D and H

It is known that the only 2-connected $\{K_{1,3}, Z_3\}$-free nonhamiltonian graphs have 9 vertices (see [25]), hence for $n \geq 10$, every 2-connected $\{K_{1,3}, Z_3\}$-free graph on n vertices is also hamiltonian. But this is not true for $\{K_{1,3}, Z_3\}$-heavy graphs. A counterexample is illustrated in Figure 6.4. We omit the details.

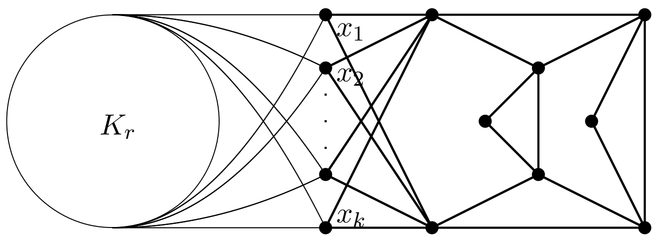


Figure 6.4: A 2-connected $\{K_{1,3}, Z_3\}$-heavy nonhamiltonian graph ($k \geq 7$, $r \geq k + 4$)

Instead of Theorems 6.3 and 6.4, we prove the following two stronger results. We refer to Figure 6.5 for an illustration of the relevant graphs $N_{1,1,2}$ and $H_{1,1}$ in the next two results.

Theorem 6.8. *If G is a 2-connected $\{K_{1,3}, N_{1,1,2}, D\}$-heavy graph, then G is* *hamiltonian.*

Theorem 6.9. *If G is a 2-connected $\{K_{1,3}, N_{1,1,2}, H_{1,1}\}$-heavy graph, then G is hamiltonian.*

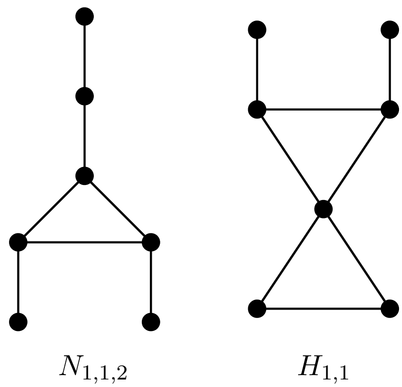


Figure 6.5: Graphs $N_{1,1,2}$ and $H_{1,1}$

Since a W-heavy graph is also $\{N_{1,1,2}, D\}$-heavy, Theorem 6.3 can be deduced from Theorem 6.8. Similarly, since an N-heavy graph is also $\{N_{1,1,2}, H_{1,1}\}$-heavy, Theorem 6.4 can be deduced from Theorem 6.9.

Note that Brousek [14] gave a complete characterization of triples of connected graphs $K_{1,3}, X, Y$ such that a graph G being 2-connected and $\{K_{1,3}, X, Y\}$-free implies G is hamiltonian. Clearly, if $K_{1,3}, S, T$ is a triple such that every 2-connected $\{K_{1,3}, S, T\}$-heavy graph is hamiltonian, then, for some triple $K_{1,3}, X, Y$ of [14], S and T are induced subgraphs of X and Y, respectively (of course, the triples of Theorems 6.8 and 6.9 have this property). We refer the interested reader to [14] for more details.

6.2 Some preliminaries

We first give some additional terminology and notation.

Let G be a graph and let X be a subset of $V(G)$. The subgraph of G

induced by X is denoted by $G[X]$. We use $G - X$ to denote the subgraph induced by $V(G) \setminus X$.

Throughout this paper, k and ℓ will always denote positive integers, and we use s and t to denote integers which may be nonpositive. For $s \leq t$, we use $[x_s, x_t]$ to denote the set $\{x_s, x_{s+1}, \ldots, x_t\}$. If $[x_s, x_t]$ is a subset of the vertex set of a graph G, we use $G[x_s, x_t]$, instead of $G[[x_s, x_t]]$, to denote the subgraph induced by $[x_s, x_t]$ in G.

For a path P and $x, y \in V(P)$, $P[x, y]$ denotes the subpath of P from x to y. Similarly, for a cycle C with a given orientation and $x, y \in V(C)$, $\overrightarrow{C}[x, y]$ or $\overleftarrow{C}[y, x]$ denotes the (x, y)-path on C traversed in the same or opposite direction with respect to the given orientation of C.

Let G be a graph and $x_1, x_2, y_1, y_2 \in V(G)$ with $x_1 \neq x_2$ and $y_1 \neq y_2$. We define an $(\{x_1, x_2\}, \{y_1, y_2\})$-*disjoint path pair*, or briefly an (x_1x_2, y_1y_2)-*pair*, as a union of two vertex-disjoint paths P and Q such that

(1) the origins of P and Q are in $\{x_1, x_2\}$, and

(2) the termini of P and Q are in $\{y_1, y_2\}$.

If G is a graph on $n \geq 2$ vertices, $x \in V(G)$, and a graph G' is obtained from G by adding a (new) vertex y and a pair of edges yx, yz, where z is an arbitrary vertex of G, $z \neq x$, we say that G' is a *1-extension of* G *at* x *to* y. Similarly, if $x_1, x_2 \in V(G)$, $x_1 \neq x_2$, then the graph G' obtained from G by adding two (new) vertices y_1, y_2 and the edges y_1x_1, y_2x_2 and y_1y_2 is called the *2-extension of* G *at* (x_1, x_2) *to* (y_1, y_2).

Let G be a graph and let $u, v, w \in V(G)$ be distinct vertices of G. We say that G is (u, v, w)-*composed* (or briefly *composed*) if G has a spanning subgraph D (called the *carrier* of G) such that there is an ordering $v_{-k}, \ldots, v_0, \ldots, v_\ell$ $(k, \ell \geq 1)$ of $V(D)$ $(=V(G))$ and a sequence of graphs $D_1, \ldots, D_r$ $(r \geq 1)$ such that

(1) $u = v_{-k}$, $v = v_0$, $w = v_\ell$,

(2) D_1 is a triangle with $V(D_1) = \{v_{-1}, v_0, v_1\}$,

(3) $V(D_i) = [v_{-k_i}, v_{\ell_i}]$ for some k_i, ℓ_i, $1 \leq k_i \leq k$, $1 \leq \ell_i \leq \ell$, and D_{i+1}, $1 \leq i \leq r-1$, satisfies one of the following:

  (*a*) D_{i+1} is a 1-extension of D_i at v_{-k_i} to v_{-k_i-1} or at v_{ℓ_i} to v_{ℓ_i+1},

  (*b*) D_{i+1} is a 2-extension of D_i at (v_{-k_i}, v_{ℓ_i}) to $(v_{-k_i-1}, v_{\ell_i+1})$,

(4) $D_r = D$.

The ordering $v_{-k}, \ldots, v_0, \ldots, v_\ell$ will be called a *canonical ordering* and the sequence $D_1, \ldots, D_r$ a *canonical sequence* of D (and also of G). Note that a composed graph G can have several carriers, canonical orderings and canonical sequences. Clearly, a composed graph G and any of its carriers D are 2-connected; moreover, for any canonical ordering, $P = v_{-k} \cdots v_0 \cdots v_\ell$ is a Hamilton path in D (called a *canonical path*), and if $D_1, \ldots, D_r$ is a canonical sequence, then any D_i is $(v_{-k_i}, v_0, v_{\ell_i})$-composed, $i = 1, \ldots, r$. Note that a (u, v, w)-composed graph is also (w, v, u)-composed.

The following lemma on composed graphs will be needed in our proofs.

Lemma 1. *Let G be a composed graph and let D and $v_{-k}, \ldots, v_0, \ldots, v_\ell$ be a carrier and a canonical ordering of G. Then*

(1) *D has a Hamilton (v_0, v_{-k})-path, and*

(2) *for every $v_s \in V(G) \setminus \{v_{-k}\}$, D has a spanning $(v_0 v_\ell, v_s v_{-k})$-pair.*

Proof. Let $D_1, \ldots, D_r$ be a canonical sequence and Q the canonical path of D corresponding to the given ordering and, for every $s \in [-k, \ell] \setminus \{0\}$, let $\hat{s}$, $1 \leq \hat{s} \leq r$, be the smallest integer for which $v_s \in V(D_{\hat{s}})$. Clearly, $d_{D_{\hat{s}}}(v_s) = 2$.

We prove (1) by induction on $|V(D)|$. If $|V(D)| = 3$, the assertion is trivially true. Suppose now that $|V(D)| \geq 4$ and that the assertion is true for every graph with at most $|V(D)| - 1$ vertices. By the definition of a carrier, we have the following two cases.

Case 1. $V(D_{r-1}) = [v_{-k+1}, v_\ell]$ and D is a 1-extension of D_{r-1} at v_{-k+1} to v_{-k}.

By the induction hypothesis, D_{r-1} has a Hamilton (v_0, v_{-k+1})-path P'. Then $P = v_0 P' v_{-k+1} v_{-k}$ is a Hamilton (v_0, v_{-k})-path in D.

Case 2. $V(D_{r-1}) = [v_{-k}, v_{\ell-1}]$ and D is a 1-extension of D_{r-1} at $v_{\ell-1}$ to v_ℓ, or $V(D_{r-1}) = [v_{-k+1}, v_{\ell-1}]$ and D is a 2-extension of D_{r-1} at $(v_{-k+1}, v_{\ell-1})$ to (v_{-k}, v_ℓ).

In this case, v_ℓ has a neighbor v_s other than $v_{\ell-1}$, where $s \in [-k, \ell - 2]$. We distinguish three subcases.

Case 2.1. $s \in [-k, -2]$.

In this case $s + 1 \in [-k + 1, -1]$. Consider the graph $D' = D_{\widehat{s+1}}$. Let $V(D') = [v_{s+1}, v_t]$, where $t > 0$. By the induction hypothesis, there exists a

Hamilton (v_0, v_t)-path P' of D'. Then the path $P = P'Q[v_t, v_\ell]v_\ell v_s Q[v_s, v_{-k}]$ is a Hamilton (v_0, v_{-k})-path of D.

Case 2.2. $s = -1$.

In this case, the path $P = Q[v_0, v_\ell]v_\ell v_{-1}Q[v_{-1}, v_{-k}]$ is a Hamilton (v_0, v_{-k})-path of D.

Case 2.3. $s \in [0, \ell - 2]$.

In this case $s+1 \in [1, \ell-1]$. Consider the graph $D' = D_{\widehat{s+1}}$. Let $V(D') = [v_t, v_{s+1}]$, where $t < 0$ and $d_{D'}(v_{s+1}) = 2$. By the induction hypothesis, there exists a Hamilton (v_0, v_t)-path P' of D', and the edge $v_s v_{s+1}$ is in $E(P')$ by the fact $d_{D'}(v_{s+1}) = 2$. Thus the path $P = P' - v_s v_{s+1} \cup Q[v_{s+1}, v_l]v_l v_s \cup Q[v_t, v_{-k}]$ is a Hamilton (v_0, v_{-k})-path of G.

So the proof of (1) is complete.

Next we prove (2). We distinguish the following three cases.

Case 1. $s \in [-k+1, 0]$.

In this case, $s - 1 \in [-k, -1]$. Consider the graph $D' = D_{\widehat{s-1}}$. Let $V(D') = [v_{s-1}, v_t]$, where $t > 0$ and $d_{D'}(v_{s-1}) = 2$. By (1), there exists a Hamilton (v_0, v_t)-path P' of D' and $v_{s-1}v_s \in E(P')$. Thus $R' = P' - v_{s-1}v_s$ is a spanning (v_0v_t, v_sv_{s-1})-pair of D', and $R = R' \cup Q[v_t, v_l] \cup Q[v_{s-1}, v_{-k}]$ is a spanning (v_0v_ℓ, v_sv_{-k})-pair of D.

Case 2. $s = 1$.

In this case, $R = Q[v_0, v_{-k}] \cup Q[v_1, v_\ell]$ is a spanning (v_0v_ℓ, v_1v_{-k})-pair of D.

Case 3. $s \in [2, \ell]$.

In this case, $s - 1 \in [1, \ell - 1]$. Consider the graph $D' = D_{\widehat{s-1}}$. Let $V(D') = [v_t, v_{s-1}]$, where $t < 0$. By (1), there exists a Hamilton (v_0, v_t)-path P' of G'. Thus $P_1 = P'Q[v_t, v_{-k}]$ and $P_2 = Q[v_s, v_\ell]$ form a spanning (v_0v_ℓ, v_sv_{-k})-pair of D.

This completes the proof of Lemma 1. □

Let G be a graph on n vertices and $k \geq 3$ an integer. A sequence of vertices $C = v_1v_2 \cdots v_kv_1$ such that for all $i \in [1, k]$ either $v_iv_{i+1} \in E(G)$ or $d(v_i) + d(v_{i+1}) \geq n$ (indices are taken modulo k) is called an *Ore-cycle* or briefly, *o-cycle* of G. The *deficit* of an o-cycle C is the integer $\operatorname{def}(C) = |\{i \in [1,k] : v_i v_{i+1} \notin E(G)\}|$. Thus, a cycle is an o-cycle of deficit 0. We define an *o-path* of G similarly.

Now, we prove the following lemma on o-cycles.

Lemma 2. *Let G be a graph and let C' be an o-cycle in G. Then there is a cycle C in G such that $V(C') \subset V(C)$.*

Proof. Let C_1 be an o-cycle in G such that $V(C') \subset V(C_1)$ and $\operatorname{def}(C_1)$ is smallest possible, and suppose, to the contrary, that $\operatorname{def}(C_1) \geq 1$. Without loss of generality suppose that $C_1 = v_1 v_2 \dots v_k v_1$, where $v_1 v_k \notin E(G)$ and $d(v_1) + d(v_k) \geq n$. We use P to denote the o-path $P = v_1 v_2 \cdots v_k$.

If v_1 and v_k have a common neighbor $x \in V(G) \backslash V(P)$, then $C_2 = v_1 P v_k x v_1$ is an o-cycle in G with $V(C') \subset V(C_2)$ and $\operatorname{def}(C_2) < \operatorname{def}(C_1)$, a contradiction. Hence $N_{G-P}(v_1) \cap N_{G-P}(v_k) = \emptyset$. Then $d_P(v_1) + d_P(v_k) \geq |V(P)|$, since $d(v_1) + d(v_k) \geq n$. Thus, there exists $i \in [2, k-1]$ such that $v_i \in N_P(v_1)$ and $v_{i-1} \in N_P(v_k)$. Now $C_2 = v_1 P[v_1, v_{i-1}] v_{i-1} v_k P[v_k, v_i] v_i v_1$ is an o-cycle with $V(C') \subset V(C_2)$ and $\operatorname{def}(C_2) < \operatorname{def}(C_1)$, a contradiction. □

Note that Lemma 2 immediately implies that if P is an (x,y)-path or an o-path in G with $|V(P)|$ larger than the length of a longest cycle in G, then $xy \notin E(G)$ and $d(x) + d(y) < n$.

In the following, we denote $\widetilde{E}(G) = \{uv : uv \in E(G) \text{ or } d(u) + d(v) \geq n\}$.

Let C be a cycle in G, $x, x_1, x_2 \in V(C)$ three distinct vertices, and set $X = V(Q)$, where Q is the (x_1, x_2)-path on C containing x. We say that the pair of vertices (x_1, x_2) is *x-good on C*, if for some $j \in \{1, 2\}$ there is a vertex $x' \in X \setminus \{x_j\}$ such that

(1) there is an (x, x_{3-j})-path P such that $V(P) = X \setminus \{x_j\}$,
(2) there is an $(xx_{3-j}, x'x_j)$-pair D such that $V(D) = X$,
(3) $d(x_j) + d(x') \geq n$.

Lemma 3. *Let G be a graph, and let C be a cycle of G with a given orientation. Let $x, y \in V(C)$ and let R be an (x,y)-path in G which is internally-disjoint from C. If there are vertices $x_1, x_2, y_1, y_2 \in V(C) \setminus \{x, y\}$ such that*

(1) *x_2, x, x_1, y_1, y, y_2 appear in this order along $\overrightarrow{C}$ (possibly $x_1 = y_1$ or $x_2 = y_2$),*
(2) *(x_1, x_2) is x-good on C,*

(3) (y_1, y_2) *is* y*-good on* C,
then there is a cycle C' *in* G *such that* $V(C) \cup V(R) \subset V(C')$.

Proof. Assume the opposite. Let P_1 and D_1 be the path and disjoint path pair associated with x, and P_2 and D_2 associated with y; and let $Q_1 = \overrightarrow{C}[x_1, y_1]$ and $Q_2 = \overleftarrow{C}[x_2, y_2]$.

By the definition of an x-good pair, without loss of generality, we can assume that P_1 is an (x, x_1)-path, D_1 is an $(xx_1, x'x_2)$-pair, and $d(x_2)+d(x') \geq n$. We distinguish a number of cases and subcases.

Case 1. P_2 is a (y, y_1)-path, D_2 is a $(yy_1, y'y_2)$-pair, and $d(y_2) + d(y') \geq n$.

In this case the path $P = Q_2 \cup D_2 \cup R \cup P_1 \cup Q_1$ is an (x_2, y')-path containing all the vertices of $V(C) \cup V(R)$, and $P' = Q_2 \cup D_1 \cup R \cup P_2 \cup Q_1$ is an (x', y_2)-path containing all the vertices of $V(C) \cup V(R)$. Thus, by Lemma 2, $d(x_2)+d(y') < n$ and $d(x')+d(y_2) < n$, a contradiction to $d(x_2)+d(x') \geq n$ and $d(y_2) + d(y') \geq n$.

Case 2. P_2 is a (y, y_2)-path, D_2 is a $(yy_2, y'y_1)$-pair, and $d(y_1) + d(y') \geq n$.

Case 2.1. The $(xx_1, x'x_2)$-pair D_1 is formed by an (x, x_2)-path and an (x_1, x')-path.

In this case, the path $P = Q_2 \cup P_2 \cup R \cup P_1 \cup Q_1$ is an (x_2, y_1)-path containing all the vertices of $V(C) \cup V(R)$, and the path $P' = D_1 \cup Q_1 \cup Q_2 \cup R \cup D_2$ is an (x', y')-path containing all the vertices of $V(C) \cup V(R)$. By Lemma 2, $d(x_2) + d(y_1) < n$ and $d(x') + d(y') < n$, a contradiction.

Case 2.2. The $(xx_1, x'x_2)$-pair D_1 is formed by an (x, x')-path and an (x_1, x_2)-path.

Case 2.2.1. The $(yy_2, y'y_1)$-pair D_2 is formed by a (y, y_1)-path and a (y_2, y')-path.

This case can be proved similarly as in Case 2.1.

Case 2.2.2. The $(yy_2, y'y_1)$-pair D_2 is formed by a (y, y')-path and a (y_1, y_2)-path.

In this case, the path $P = Q_2 \cup D_2 \cup R \cup P_1 \cup Q_1$ is an (x_2, y')-path containing all the vertices of $V(C) \cup V(R)$, and the path $P' = Q_2 \cup D_1 \cup R \cup P_2 \cup Q_1$ is an (x', y_1)-path containing all the vertices of $V(C) \cup V(R)$. By Lemma 2, $d(x_2) + d(y') < n$ and $d(x') + d(y_1) < n$, a contradiction.

This completes the proof of Lemma 3. □

6.3 Proof of Theorem 6.8

Let C be a longest cycle of G. Set $n = |V(G)|$ and $c = |V(C)|$, and assume that G is not hamiltonian, i.e., $c < n$. Then $V(G)\backslash V(C) \neq \emptyset$. Since G is 2-connected, there exists a (u_0, v_0)-path with length at least 2 which is internally-disjoint from C, where $u_0, v_0 \in V(C)$. Let $R = z_0z_1z_2 \cdots z_{r+1}$, where $z_0 = u_0$ and $z_{r+1} = v_0$, be such a path, and choose R as short as possible. Let r_1 and r_2 denote the number of interior vertices in the two subpaths of C from u_0 to v_0 (note that clearly $r_1 + r_2 + 2 = c$). We specify an orientation of C, and label the vertices of C using two distinct notations u_i and v_i, $-r_2 \leq i \leq r_1$, such that $\overrightarrow{C} = u_0u_1u_2 \cdots u_{r_1}v_0u_{-r_2}u_{-r_2+1} \cdots u_{-1}u_0$ and $\overleftarrow{C} = v_0v_1v_2 \cdots v_{r_1}u_0v_{-r_2}v_{-r_2+1} \cdots v_{-1}v_0$, where $u_\ell = v_{r_1+1-\ell}$ and $u_{-k} = v_{-r_2-1+k}$ (see Figure 6.6). Let H be the component of $G - C$ containing the vertices in $[z_1, z_r]$.

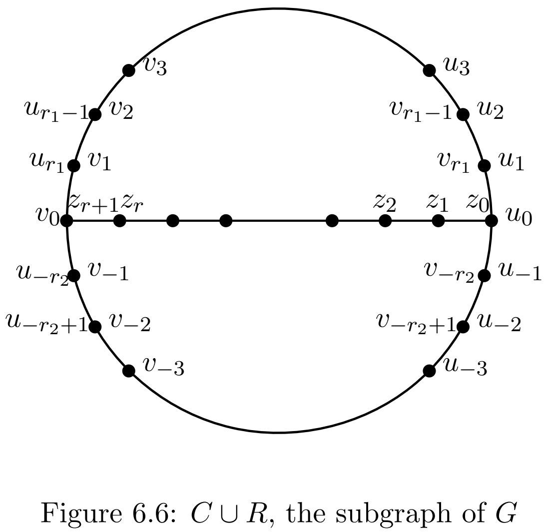


Figure 6.6: $C \cup R$, the subgraph of G

Claim 1. Let $x \in V(H)$ and $y \in \{u_1, u_{-1}, v_1, v_{-1}\}$. Then $xy \notin \widetilde{E}(G)$.

Proof. Without loss of generality, we assume $y = u_1$. Let P' be an (x, z_1)-path in H. Then $P = P' z_1 u_0 \overleftarrow{C}[u_0, u_1]$ is an (x, y)-path containing all the vertices of $V(C) \cup V(P')$. By Lemma 2, we get $xy \notin \widetilde{E}(G)$. □

Claim 2. $u_1 u_{-1} \in \widetilde{E}(G)$ and $v_1 v_{-1} \in \widetilde{E}(G)$.

Proof. If $u_1 u_{-1} \notin E(G)$, by Claim 1, the graph induced by $\{u_0, z_1, u_1, u_{-1}\}$ is a claw, where $d(z_1) + d(u_{\pm 1}) < n$. Since G is a claw-heavy graph, we get that $d(u_1) + d(u_{-1}) \geq n$.

The second assertion can be proved similarly. □

Claim 3. $u_1 v_{-1} \notin \widetilde{E}(G)$, $u_{-1} v_1 \notin \widetilde{E}(G)$, $u_0 v_{\pm 1} \notin \widetilde{E}(G), u_{\pm 1} v_0 \notin \widetilde{E}(G)$.

Proof. Since $\overrightarrow{C}[u_1, v_0] R \overleftarrow{C}[u_0, v_{-1}]$ is a (u_1, v_{-1})-path containing all the vertices of $V(C) \cup V(R)$, we obtain $u_1 v_{-1} \notin \widetilde{E}(G)$ by Lemma 2.

If $u_0 v_1 \in \widetilde{E}(G)$, then $\overrightarrow{C}[u_1, v_1] v_1 u_0 R \overrightarrow{C}[v_0, u_{-1}] u_{-1} u_1$ is an o-cycle containing all the vertices of $V(C) \cup V(R)$. By Lemma 2, there exists a cycle containing all the vertices of $V(C) \cup V(R)$, a contradiction.

The other assertions can be proved similarly. □

Claim 4. Either $u_1 u_{-1} \in E(G)$ or $v_1 v_{-1} \in E(G)$.

Proof. Assume the opposite. By Claim 2, $d(u_1) + d(u_{-1}) \geq n$ and $d(v_1) + d(v_{-1}) \geq n$. By Claim 3, $d(u_1) + d(v_{-1}) < n$ and $d(u_{-1}) + d(v_1) < n$, a contradiction. □

Now, we distinguish two cases, namely the case that $r \geq 2$, or $r = 1$ and $u_0 v_0 \notin E(G)$, and the case that $r = 1$ and $u_0 v_0 \in E(G)$.

Case 1. $r \geq 2$, or $r = 1$ and $u_0 v_0 \notin E(G)$.

By Claim 4, without loss of generality, we assume that $u_1 u_{-1} \in E(G)$. Thus $G[u_{-1}, u_1]$ is (u_{-1}, u_0, u_1)-composed.

Claim 5. $z_2 u_0 \notin \widetilde{E}(G)$.

Proof. By the choice of the path R, $z_2 u_0 \notin E(G)$. Now we prove that $d(z_2) + d(u_0) < n$.

Claim 5.1. Every neighbor of u_0 is in $V(C) \cup V(H)$; every neighbor of z_2 is in $V(C) \cup V(H)$.

Proof. Assume the opposite. Let $z' \in V(H')$ be a neighbor of u_0 where H' is a component of $G - C$ other than H. Then $z'z_1 \notin E(G)$ and $N_{G-C}(z') \cap N_{G-C}(z_1) = \emptyset$.

By Claim 1, $u_1 z_1 \notin \widetilde{E}(G)$, and similarly $u_1 z' \notin \widetilde{E}(G)$. Thus the graph induced by $\{u_0, u_1, z_1, z'\}$ is a claw, where $d(u_1)+d(z_1) < n$ and $d(u_1)+d(z') < n$. Hence, $d(z_1) + d(z') \geq n$.

Since $N_{G-C}(z_1) \cap N_{G-C}(z') = \emptyset$, there exist two vertices $x_1, x_2 \in V(C)$ such that $x_1 x_2 \in E(\overrightarrow{C})$ and $z_1 x_1, z' x_2 \in E(G)$. Then $z_1 x_1 \overleftarrow{C}[x_1, x_2] x_2 z'$ is a (z_1, z')-path containing all the vertices of $V(C) \cup \{z_1, z'\}$. By Lemma 2, there exists a cycle containing all the vertices of $V(C) \cup \{z_1, z'\}$, a contradiction.

If $z_2 = v_0$, the second assertion can be proved similarly; and if $z_2 \neq v_0$, the assertion is obvious. □

Let $h = |V(H)|$ and $k = |N_H(u_0)|$. Then $d_H(z_2) + d_H(u_0) \leq h + k$. Since $z_1 \in N_H(u_0)$, $k \geq 1$. Let $N_H(u_0) = \{y_1, y_2, \ldots, y_k\}$, where $y_1 = z_1$.

Claim 5.2. $y_i y_j \in \widetilde{E}(G)$ for all $1 \leq i < j \leq k$.

Proof. If $y_i y_j \notin E(G)$, then by Claim 1, the graph induced by $\{u_0, u_1, y_i, y_j\}$ is a claw, where $d(y_i) + d(u_1) < n$ and $d(y_j) + d(u_1) < n$. Thus $d(y_i) + d(y_j) \geq n$. □

Now, let Q be the o-path $Q = z_2 y_1 y_2 \cdots y_k u_0$. It is clear that $R[z_2, v_0]$ and Q are internally-disjoint, and Q contains at least k vertices of $V(H)$. In the following, we use C' to denote the cycle $\overrightarrow{C}[u_1, u_{-1}] u_{-1} u_1$ if $z_2 \neq v_0$, and to denote the o-cycle $\overrightarrow{C}[u_1, v_1] v_1 v_{-1} \overrightarrow{C}[v_{-1}, u_{-1}] u_{-1} u_1$ if $z_2 = v_0$.

By Claims 1 and 3, $z_2 v_{r_1} \notin E(G)$, where $v_{r_1} = u_1$. Let v_ℓ be the last vertex in $\overleftarrow{C}[v_1, u_1]$ such that $z_2 v_\ell \in E(G)$. If there are no neighbors of z_2 in $\overleftarrow{C}[v_1, u_1]$, then let $v_\ell = v_0$.

Claim 5.3. For every vertex $v_{\ell'} \in N_{[v_1, v_{r_1}]}(z_2) \cup \{v_0\}$, $u_0 v_{\ell'+1} \notin E(G)$.

Proof. By Claim 3, $u_0 v_1 \notin E(G)$.

If $z_2 v_{\ell'} \in E(G)$ and $u_0 v_{\ell'+1} \in E(G)$, then $\overrightarrow{C'}[v_{\ell'}, v_{\ell'+1}] v_{\ell'+1} u_0 Q z_2 v_{\ell'}$ is an o-cycle containing all the vertices of $V(C) \cup V(Q)$, a contradiction. □

Claim 5.4. $r_1-\ell \geq k+1$, and for every vertex $v_{\ell'} \in [v_{\ell+1}, v_{\ell+k}]$, $u_0v_{\ell'} \notin E(G)$.

Proof. Assume the opposite. Let $v_{\ell'}$ be the first vertex in $[v_{\ell+1}, v_{r_1}]$ such that $u_0v_{\ell'} \in E(G)$, and $\ell' - \ell < k+1$.

If $v_\ell = v_0$, then $C'' = \overrightarrow{C}[v_0, u_{-1}]u_{-1}u_1\overrightarrow{C}[u_1, v_{\ell'}]v_{\ell'}u_0QR[z_2, v_0]$ is an o-cycle containing all the vertices of $V(C)\backslash[v_1, v_{\ell'-1}] \cup V(Q)$, and $|V(C'')| > c$, a contradiction.

Thus, we assume that $v_\ell \neq v_0$, and $z_2v_\ell \in E(G)$. Then $C'' = \overrightarrow{C'}[v_\ell, v_{\ell'}]v_{\ell'}u_0 Qz_2v_\ell$ is an o-cycle containing all the vertices of $V(C)\backslash[v_{\ell+1}, v_{\ell'-1}]\cup V(Q)$, and $|V(C'')| > c$, a contradiction.

Thus $\ell' - \ell \geq k+1$. Noting that $u_0v_{r_1} \in E(G)$, we get $r_1 - \ell \geq k+1$. □

Let $d_1 = |N_{[v_1,v_{r_1}]}(z_2)\cup\{v_0\}|$, $d_2 = |N_{[v_{-r_2},v_{-1}]}(z_2)\cup\{v_0\}|$, $d_1' = |N_{[v_1,v_{r_1}]}(u_0)|$ and $d_2' = |N_{[v_{-r_2},v_{-1}]}(u_0)|$. Then $d_C(z_2) \leq d_1+d_2-1$ and $d_C(u_0) \leq d_1'+d_2'+1$.

By Claims 5.3 and 5.4, we have $d_1' \leq r_1 - d_1 - k + 1$, and similarly, $d_2' \leq r_2 - d_2 - k + 1$. Thus $d_C(z_2) + d_C(u_0) \leq r_1 + r_2 - 2k + 2 = c - 2k$. Note that $d_H(z_2)+d_H(u_0) \leq h+k$. By Claim 5.1, $d(z_2)+d(u_0) \leq c+h-k < n$. □

Recall that $G[u_{-1}, u_1]$ is (u_{-1}, u_0, u_1)-composed. Now we prove the following claims.

Claim 6. If $G[u_{-k}, u_\ell]$ is (u_{-k}, u_0, u_ℓ)-composed with canonical ordering u_{-k}, $u_{-k+1}, \ldots, u_\ell$, then $k \leq r_2 - 2$ and $\ell \leq r_1 - 2$.

Proof. Let $D_1, D_2, \ldots, D_r$ be a canonical sequence of $G[u_{-k}, u_\ell]$ corresponding to the canonical ordering $u_{-k}, u_{-k+1}, \ldots, u_\ell$. Suppose that $k > r_2 - 2$. Consider the graph $D' = D_{\widehat{-r_2+1}}$, where $\widehat{-r_2+1}$ is the smallest integer such that $u_{-r_2+1} \in V(D_{\widehat{-r_2+1}})$. Let $V(D') = [u_{-r_2+1}, u_{\ell'}]$. By Lemma 1, there exists a $(u_0, u_{\ell'})$-path P such that $V(P) = [u_{-r_2+1}, u_{\ell'}]$. Then $v_{-1}v_0RP\overrightarrow{C}[u_{\ell'}, v_1]v_1v_{-1}$ is an o-cycle containing all the vertices of $V(C) \cup V(R)$, a contradiction.

Similarly, we can prove that $\ell \leq r_1 - 2$. □

Claim 7. If $G[u_{-k}, u_\ell]$ is (u_{-k}, u_0, u_ℓ)-composed with canonical ordering u_{-k}, $u_{-k+1}, \ldots, u_\ell$, where $k \leq r_2 - 2$ and $l \leq r_1 - 2$, and any two nonadjacent vertices in $[u_{-k-1}, u_{\ell+1}]$ have degree sum less than n, then one of the following is true:

(1) $G[u_{-k-1}, u_\ell]$ is (u_{-k-1}, u_0, u_ℓ)-composed with canonical ordering u_{-k-1}, $u_{-k}, \ldots, u_\ell$,

(2) $G[u_{-k}, u_{\ell+1}]$ is $(u_{-k}, u_0, u_{\ell+1})$-composed with canonical ordering u_{-k}, $u_{-k+1}, \ldots, u_{\ell+1}$, or
(3) $G[u_{-k-1}, u_{\ell+1}]$ is $(u_{-k-1}, u_0, u_{\ell+1})$-composed with canonical ordering $u_{-k-1}, u_{-k}, \ldots, u_{\ell+1}$.

Proof. Assume the opposite, which implies that for every vertex $u_s \in [u_{-k+1}, u_\ell]$, $u_{-k-1}u_s \notin E(G)$, and for every vertex $u_s \in [u_{-k}, u_{\ell-1}]$, $u_{\ell+1}u_s \notin E(G)$ and $u_{-k-1}u_{\ell+1} \notin E(G)$.

Claim 7.1. Let $z \in \{z_1, z_2\}$ and $u_s \in [u_{-k-1}, u_{\ell+1}]\backslash\{u_0\}$. Then $zu_s \notin \widetilde{E}(G)$.

Proof. Without loss of generality, we assume that $s > 0$. If $s = 1$, the assertion is true by Claims 1 and 3. So we assume that $s \in [2, \ell+1]$ and $s-1 \in [1, \ell]$. By the definition of a composed graph, there exists $t \in [-k, -1]$ such that $G[u_t, u_{s-1}]$ is (u_t, u_0, u_{s-1})-composed. By Lemma 1, there exists a (u_0, u_t)-path P' such that $V(P') = [u_t, u_{s-1}]$.

If $z \neq v_0$, then $R[z, u_0]P'\overleftarrow{C}[u_t, u_s]$ is a (z, u_s)-path containing all the vertices of $V(C) \cup \{z\}$. By Lemma 2, $zu_s \notin \widetilde{E}(G)$.

If $z = v_0$ and $v_0u_s \in \widetilde{E}(G)$, then $RP'\overleftarrow{C}[u_t, v_{-1}]v_{-1}v_1\overleftarrow{C}[v_1, u_s]u_sv_0$ is an o-cycle containing all the vertices of $V(C) \cup V(R)$, a contradiction. □

Let $G' = G[[u_{-k-1}, u_\ell] \cup \{z_1, z_2\}]$ and $G'' = G[[u_{-k-1}, u_{\ell+1}] \cup \{z_1, z_2\}]$.

Claim 7.2. G'', and hence G', are $\{K_{1,3}, N_{1,1,2}\}$-free.

Proof. By Claims 5 and 7.1, and the condition that any two nonadjacent vertices in $[u_{-k-1}, u_{\ell+1}]$ have degree sum less than n, any two nonadjacent vertices in G'' have degree sum less than n. Since G (and hence G'') is $\{K_{1,3}, N_{1,1,2}\}$-heavy, G'' is $\{K_{1,3}, N_{1,1,2}\}$-free. □

Claim 7.3. $N_{G'}(u_0)\backslash\{z_1\}$ is a clique.

Proof. If there are two vertices $x, x' \in N_{G'}(u_0)\backslash\{z_1\}$ such that $xx' \notin E(G')$, then the graph induced by $\{u_0, z_1, x, x'\}$ is a claw, a contradiction. □

Now, we define $N_i = \{x \in V(G') : d_{G'}(x, u_{-k-1}) = i\}$. Then $N_0 = \{u_{-k-1}\}$, $N_1 = \{u_{-k}\}$ and $N_2 = N_{G'}(u_{-k})\backslash\{u_{-k-1}\}$.

By the definition of a composed graph, $|N_2| \geq 2$. If there are two vertices $x, x' \in N_2$ such that $xx' \notin E(G')$, then the graph induced by $\{u_{-k}, u_{-k-1}, x, x'\}$ is a claw, a contradiction. Thus, N_2 is a clique.

We assume $u_0 \in N_j$, where $j \geq 2$. Then $z_1 \in N_{j+1}$ and $z_2 \in N_{j+2}$.

If $|N_i| = 1$ for some $i \in [2, j-1]$, say, $N_i = \{x\}$, then x is a cut vertex of the graph $G[u_{-k}, u_l]$. By the definition of a composed graph, $G[u_{-k}, u_l]$ is 2-connected. This implies $|N_i| \geq 2$ for every $i \in [2, j-1]$.

Claim 7.4. For $i \in [1, j]$, N_i is a clique.

Proof. We prove this claim by induction on i. For $i = 1, 2$, the claim is true by the above analysis. So we assume that $3 \leq i \leq j$, and $N_{i-3}, N_{i-2}, N_{i-1}, N_{i+1}$ and N_{i+2} are nonempty, and $|N_{i-1}| \geq 2$.

First we choose a vertex $x \in N_i$ which has a neighbor $y \in N_{i+1}$ such that it has a neighbor $z \in N_{i+2}$. We prove that for every $x' \in N_i$, $xx' \in E(G)$. To the contrary, we assume that $xx' \notin E(G)$.

If $x'y \in E(G)$, then the graph induced by $\{y, x, x', z\}$ is a claw, a contradiction. Thus, $x'y \notin E(G)$. If x and x' have a common neighbor in N_{i-1}, denote it by w; then let v be a neighbor of w in N_{i-2}, and the graph induced by $\{w, v, x, x'\}$ is a claw, a contradiction. Thus x and x' have no common neighbors in N_{i-1}.

Let w be a neighbor of x in N_{i-1}, and let w' be a neighbor of x' in N_{i-1}. Then $xw', x'w \notin E(G)$. Let v be a neighbor of w in N_{i-2} and u be a neighbor of v in N_{i-3}. If $w'v \notin E(G)$, then the graph induced by $\{w, v, w', x\}$ is a claw, a contradiction. Thus $w'v \in E(G)$, and then the graph induced by $\{v, u, w', x', w, x, y\}$ is an $N_{1,1,2}$, a contradiction.

Thus $xx' \in E(G)$ for every $x' \in N_i$, as we claimed.

Now, let x' and x'' be two vertices in N_i other than x such that $x'x'' \notin E(G)$. We have $xx', xx'' \in E(G)$.

If $x'y \in E(G)$, then similarly to the case of x, we have $x'x'' \in E(G)$, a contradiction. Thus $x'y \notin E(G)$. Similarly, $x''y \notin E(G)$. Then the graph induced by $\{x, x', x'', y\}$ is a claw, a contradiction.

We conclude that N_i is a clique. □

If there exists some vertex $y \in N_{j+1}$ other than z_1, then $yu_0 \notin E(G)$ by Claim 7.3. Let x be a neighbor of y in N_j, let w be a neighbor of u_0 in N_{j-1}, and let v be a neighbor of w in N_{j-2}. Then $xu_0 \in E(G)$ by Claim 7.4 and $xw \in E(G)$ by Claim 7.3. Thus the graph induced by $\{w, v, x, y, u_0, z_1, z_2\}$ is an $N_{1,1,2}$, a contradiction. So we assume that all vertices in $[u_{-k}, u_\ell]$ are in $\bigcup_{i=1}^{j} N_i$.

If $u_\ell \in N_j$, then let w be a neighbor of u_0 in N_{j-1}, and let v be a neighbor of w in N_{j-2}. Then the graph induced by $\{w, v, u_0, z_1, u_\ell, u_{\ell+1}\}$ is an $N_{1,1,2}$, a contradiction. Thus $u_\ell \notin N_j$ and $j \geq 3$.

Let $u_\ell \in N_i$, where $i \in [2, j-1]$. If u_ℓ has a neighbor in N_{i+1}, then let y be a neighbor of u_ℓ in N_{i+1}, and let w be a neighbor of u_ℓ in N_{i-1}. Then the graph induced by $\{u_\ell, w, y, u_{\ell+1}\}$ is a claw, a contradiction. So u_ℓ has no neighbors in N_{i+1}.

Let $x \in N_i$ be a vertex other than u_ℓ that has a neighbor y in N_{i+1} such that it has a neighbor z in N_{i+2}. Let w be a neighbor of x in N_{i-1}, and let v be a neighbor of w in N_{i-2}. If $u_\ell w \notin E(G)$, then the graph induced by $\{x, w, u_\ell, y\}$ is a claw, a contradiction. So $u_\ell w \in E(G)$. Then the graph induced by $\{w, v, u_\ell, u_{\ell+1}, x, y, z\}$ is an $N_{1,1,2}$, a contradiction.

This completes the proof of Claim 7. □

Now we choose k, ℓ such that

(1) $G[u_{-k}, u_\ell]$ is (u_{-k}, u_0, u_ℓ)-composed with canonical ordering $u_{-k}, u_{-k+1}, \ldots, u_\ell$;

(2) any two nonadjacent vertices in $[u_{-k}, u_\ell]$ have degree sum less than n; and

(3) $k + \ell$ is as large as possible.

By Claim 7, there exists a vertex $u_s \in [u_{-k+1}, u_\ell]$ such that $d(u_{-k-1}) + d(u_s) \geq n$, or there exists a vertex $u_s \in [u_{-k}, u_{\ell-1}]$ such that $d(u_s) + d(u_{\ell+1}) \geq n$, or $d(u_{-k-1}) + d(u_{\ell+1}) \geq n$.

Claim 8. (u_{-k-1}, u_ℓ) or $(u_{-k}, u_{\ell+1})$ or $(u_{-k-1}, u_{\ell+1})$ is u_0-good on C.

Proof. If there exists a vertex $u_s \in [u_{-k+1}, u_\ell]$ such that $d(u_{-k-1}) + d(u_s) \geq n$, then, by Lemma 1, there exists a (u_0, u_ℓ)-path P such that $V(P) = [u_{-k}, u_\ell]$, and there exists a $(u_0 u_\ell, u_s u_{-k})$-pair D' such that $V(D') = [u_{-k}, u_\ell]$. Then $D = D' + u_{-k} u_{-k-1}$ is a $(u_0 u_\ell, u_s u_{-k-1})$-pair such that $V(D) = [u_{-k-1}, u_\ell]$. Thus (u_{-k-1}, u_ℓ) is u_0-good on C.

If there exists a vertex $u_s \in [u_{-k}, u_{\ell-1}]$ such that $d(u_s) + d(u_{\ell+1}) \geq n$, we can prove the result similarly.

If $d(u_{-k-1}) + d(u_{\ell+1}) \geq n$, then by Lemma 1, there exists a (u_0, u_ℓ)-path P' such that $V(P') = [u_{-k}, u_\ell]$ and there exists a (u_0, u_{-k})-path P'' such that $V(P'') = [u_{-k}, u_\ell]$. Then $P = P' u_1 u_{\ell+1}$ is a $(u_0, u_{\ell+1})$-path such that $V(P) = [u_{-k}, u_{\ell+1}]$, and $D = P'' u_{-k} u_{-k-1} \cup u_{\ell+1}$ is a $(u_0 u_{\ell+1}, u_{\ell+1} u_{-k-1})$-pair such that $V(D) = [u_{-k-1}, u_{\ell+1}]$. Thus $(u_{-k-1}, u_{\ell+1})$ is u_0-good on C. □

Claim 9. There exist $v_{-k'} \in V(\overrightarrow{C}[v_{-1}, u_{-k-1}])$ and $v_{\ell'} \in V(\overleftarrow{C}[v_1, u_{\ell+1}])$ such that $(v_{-k'}, v_{\ell'})$ is v_0-good on C.

Proof. By Claim 6, $k \leq r_2 - 2$ and $l \leq r_1 - 2$.

If $v_1v_{-1} \notin E(G)$, then by Claim 2, $d(v_1) + d(v_{-1}) \geq n$. Then $P = v_0v_1$ is a (v_0, v_1)-path and $D = v_0v_{-1} \cup v_1$ is a $(v_0v_1, v_{-1}v_1)$-pair. Then (v_{-1}, v_1) is v_0-good on C.

Now we assume that $v_1v_{-1} \in E(G)$, and then $G[v_{-1}, v_1]$ is (v_{-1}, v_0, v_1)-composed.

Let $r_2' = r_2 - k$ and $r_1' = r_1 - \ell$.

Claim 9.1. If $G[v_{-k'}, v_{\ell'}]$ is $(v_{-k'}, v_0, v_{\ell'})$-composed with canonical ordering $v_{-k'}, v_{-k'+1}, \ldots, v_{\ell'}$, then $k' \leq r_2' - 1$ and $\ell' \leq r_1' - 1$.

Proof. Let $D_1, D_2, \ldots, D_r$ be a canonical sequence of $G[v_{-k'}, v_{\ell'}]$ corresponding to the canonical ordering $v_{-k'}, v_{-k'+1}, \ldots, v_{\ell'}$. Suppose that $k' > r_2' - 1$. Consider the graph $D' = D_{\widehat{-r_2'}}$, where $\widehat{-r_2'}$ is the smallest integer such that $v_{-r_2'} \in V(D_{\widehat{-r_2'}})$. Let $V(D') = [v_{-r_2'}, v_{\ell''}]$. By Lemma 1, there exists a $(v_0, v_{\ell''})$-path P such that $V(P) = [v_{-r_2'}, u_{\ell''}]$. Then $P\overrightarrow{C}[u_\ell, v_{\ell''}]P'R$ is a cycle containing all the vertices of $V(C) \cup V(R)$, a contradiction. □

Similar to Claim 7, we obtain the following claim that we present without proof.

Claim 9.2. If $G[v_{-k'}, v_{\ell'}]$ is $(v_{-k'}, v_0, v_{\ell'})$-composed with canonical ordering $v_{-k'}, v_{-k'+1}, \ldots, v_{\ell'}$, where $k' \leq r_2' - 1$ and $\ell \leq r_1' - 1$, and any two nonadjacent vertices in $[v_{-k'-1}, v_{\ell'+1}]$ have degree sum less than n, then one of the following is true:

(1) $G[v_{-k'-1}, v_{\ell'}]$ is $(v_{-k'-1}, v_0, v_{\ell'})$-composed with canonical ordering $v_{-k'-1}, v_{-k'}, \ldots, v_{\ell'}$,

(2) $G[v_{-k'}, v_{l'+1}]$ is $(v_{-k'}, v_0, v_{\ell'+1})$-composed with canonical ordering $v_{-k'}, v_{-k'+1}, \ldots, v_{\ell'+1}$, or

(3) $G[v_{-k'-1}, v_{l'+1}]$ is $(v_{-k'-1}, v_0, v_{\ell'+1})$-composed with canonical ordering $v_{-k'-1}, v_{-k'}, \ldots, v_{\ell'+1}$.

Now we choose k', ℓ' such that

(1) $G[v_{-k'}, v_{\ell'}]$ is $(v_{-k'}, v_0, v_{\ell'})$-composed with canonical ordering $v_{-k'}, v_{-k'+1}, \ldots, v_{\ell'}$;

(2) any two nonadjacent vertices in $[v_{-k'}, v_{\ell'}]$ have degree sum less than n; and

(3) $k' + \ell'$ is as large as possible.

Similar as in Claim 8, we obtain that $(v_{-k'-1},\ v_{l'})$ or $(v_{-k'}, v_{l'+1})$ or $(v_{-k'-1}, v_{l'+1})$ is v_0-good on C. This completes the proof of Claim 9. □

From Claims 8 and 9, we get that there exists a cycle containing all the vertices of $V(C) \cup V(R)$ by Lemma 3, a contradiction. This completes Case 1.

Case 2. $r = 1$ and $u_0v_0 \in E(G)$.

We have $u_0u_{-1} \in E(G)$ and $u_0u_{-r_2} \notin E(G)$, where $u_{-r_2} = v_{-1}$. Let u_{-k-1} be the first vertex in $\overleftarrow{C}[u_{-1}, v_{-1}]$ such that $u_0u_{-k-1} \notin E(G)$. Then $k \le r_2 - 1$.

Similarly, let $v_{\ell+1}$ be the first vertex in $\overleftarrow{C}[v_1, u_1]$ such that $v_0v_{\ell+1} \notin E(G)$. Then $\ell \le r_1 - 1$.

Claim 10. Let $x \in [u_{-k-1}, u_{-1}]$ and $y \in [v_1, v_{\ell+1}]$. Then

(1) $xz_1, xv_0 \notin \widetilde{E}(G)$,

(2) $yz_1, yu_0 \notin \widetilde{E}(G)$,

(3) $xy \notin \widetilde{E}(G)$.

Proof. (1) If $x = u_{-1}$, then by Claims 1 and 3, $u_{-1}z_1, u_{-1}v_0 \notin \widetilde{E}(G)$. So we assume that $x = u_{-k'}$ where $-k' \in [-k-1, -2]$ and $u_0u_{-k'+1} \in E(G)$.

If $u_{-k'}z_1 \in \widetilde{E}(G)$, then $u_0u_{-k'+1}\overrightarrow{C}[u_{-k'+1}, u_{-1}]u_{-1}u_1\overrightarrow{C}[u_1, u_{-k'}]u_{-k'}\ z_1u_0$ is an o-cycle containing all the vertices of $V(C) \cup V(R)$, a contradiction.

If $u_{-k'}v_0 \in \widetilde{E}(G)$, then $u_0u_{-k'+1}\overrightarrow{C}[u_{-k'+1}, u_{-1}]u_{-1}u_1\overrightarrow{C}[u_1, v_1]v_1v_{-1}\ \overleftarrow{C}[v_{-1}, u_{-k'}]u_{-k'}v_0R$ is an o-cycle containing all the vertices of $V(C) \cup V(R)$, a contradiction.

The assertion (2) can be proved similarly.

(3) If $x = u_{-1}$ and $y = v_1$, then by Claim 1, $xy \notin \widetilde{E}(G)$.

If $u_{-k'}v_1 \in \widetilde{E}(G)$, where $k' \in [2, k+1]$, then $u_0R\overrightarrow{C}[v_0, u_{-k'}]u_{-k'}v_1\ \overleftarrow{C}[v_1, u_1]\ u_1u_{-1}\overleftarrow{C}[u_{-1}, u_{-k'+1}]u_{-k'+1}u_0$ is an o-cycle containing all the vertices of $V(C) \cup V(R)$, a contradiction.

If $u_{-1}v_{\ell'} \in \widetilde{E}(G)$, where $\ell' \in [2, \ell+1]$, then we can prove the result similarly.

If $u_{-k'}v_{\ell'} \in \widetilde{E}(G)$, where $k' \in [2, k+1]$ and $\ell' \in [2, \ell+1]$, then $u_0u_{-k'+1}$ $\overrightarrow{C}[u_{-k'+1}, u_{-1}]u_{-1}u_1\overrightarrow{C}[u_1, v_{l'}]v_{l'}u_{-k'}\overleftarrow{C}[u_{-k'}, v_{-1}]v_{-1}v_1\overleftarrow{C}[v_1, v_{l'-1}]v_{l'-1}\ v_0R$ is an o-cycle containing all the vertices of $V(C) \cup V(R)$, a contradiction. □

Claim 11. Either $u_{-k-1}u_0 \notin \widetilde{E}(G)$ or $v_{\ell+1}v_0 \notin \widetilde{E}(G)$.

Proof. Assume the opposite. Since $u_{-k-1}u_0, v_{\ell+1}v_0 \notin E(G)$, $d(u_{-k-1}) + d(u_0) \geq n$ and $d(v_{\ell+1}) + d(v_0) \geq n$. By Claim 10, $d(u_0) + d(v_{\ell+1}) < n$ and $d(v_0) + d(u_{-k-1}) < n$, a contradiction. □

Without loss of generality, we assume that $u_{-k-1}u_0 \notin \widetilde{E}(G)$. If $v_{\ell+1}v_0 \notin \widetilde{E}(G)$, then the subgraph induced by $\{z_1, v_0, v_\ell, v_{\ell+1}, u_0, u_{-k}, u_{-k-1}\}$ is a D which is not heavy, a contradiction. Since $v_0v_{\ell+1} \notin E(G)$, $d(v_0)+d(v_{\ell+1}) \geq n$.

Claim 12. Either (v_{-1}, v_1) or $(v_{-1}, v_{\ell+1})$ is v_0-good on C.

Proof. If $v_1v_{-1} \notin E(G)$, then, by Claim 2, $d(v_1) + d(v_{-1}) \geq n$. Then v_0v_1 is a (v_0, v_1)-path and $v_0v_{-1} \cup v_1$ is a $(v_0v_1, v_{-1}v_1)$-pair. Thus, (v_{-1}, v_1) is v_0-good on C.

If $v_1v_{-1} \in E(G)$, then $v_0v_\ell\overrightarrow{C}[v_\ell, v_1]v_1v_{-1}$ is a (v_0, v_{-1})-path and $v_0 \cup v_{-1}v_1\overleftarrow{C}[v_1, v_{\ell+1}]$ is a (v_0v_{-1}, v_0v_{l+1})-pair. Since $d(v_0)+d(v_{\ell+1}) \geq n$, $(v_{-1}, v_{\ell+1})$ is v_0-good on C. □

Claim 13. If $G[u_{-k'}, u_{\ell'}]$ is $(u_{-k'}, u_0, u_{\ell'})$-composed with canonical ordering $u_{-k'}, u_{-k'+1}, \ldots, u_{\ell'}$, then $k' \leq r_2 - 2$ and $\ell' \leq r_1 - \ell - 1$.

Proof. The claim can be proved similarly as Claims 6 and 9.1. □

Next we prove the following claim.

Claim 14. If $G[u_{-k'}, u_{\ell'}]$ is $(u_{-k'}, u_0, u_{\ell'})$-composed with canonical ordering $u_{-k'}, u_{-k'+1}, \ldots, u_{\ell'}$, where $k' \leq r_2 - 2$ and $\ell' \leq r_1 - \ell - 1$, and any two nonadjacent vertices in $[u_{-k'-1}, u_{\ell'+1}]$ have degree sum less than n, then one of the following is true:

(1) $G[u_{-k'-1}, u_{\ell'}]$ is $(u_{-k'-1}, u_0, u_{\ell'})$-composed with canonical ordering $u_{-k'-1}$, $u_{-k'}, \ldots, u_{\ell'}$,

(2) $G[u_{-k'}, u_{\ell'+1}]$ is $(u_{-k'}, u_0, u_{\ell'+1})$-composed with canonical ordering $u_{-k'}$, $u_{-k'+1}, \ldots, u_{\ell'+1}$, or

(3) $G[u_{-k'-1}, u_{\ell'+1}]$ is $(u_{-k'-1}, u_0, u_{\ell'+1})$-composed with canonical ordering $u_{-k'-1}, u_{-k'}, \ldots, u_{\ell'+1}$.

Proof. Assume the opposite, which implies that for every vertex $u_s \in [u_{-k'+1}, u_{\ell'}]$, $u_{-k'-1}u_s \notin E(G)$, and for every vertex $u_s \in [u_{-k'}, u_{\ell'-1}]$, $u_{\ell'+1}u_s \notin E(G)$, and $u_{-k'-1}u_{\ell'+1} \notin E(G)$.

Claim 14.1. Let $v \in \{v_0, v_1\}$ and $u_s \in [u_{-k'-1}, u_{\ell'+1}]\backslash\{u_0\}$. Then $vu_s \notin \widetilde{E}(G)$.

Proof. Similar to Claim 7.1, we get that $v_0u_s \notin \widetilde{E}(G)$.

Now we assume that $v_1u_s \in \widetilde{E}(G)$.

Note that if $v_0v_2 \notin E(G)$, then $d(v_0) + d(v_2) \geq n$. We have $v_0v_2 \in \widetilde{E}(G)$.

If $s \in [-k'-1, -2]$, then $s+1 \in [-k', -1]$. By the definition of a composed graph, there exists $t \in [1, \ell']$ such that $G[u_{s+1}, u_t]$ is (u_{s+1}, u_0, u_t)-composed. By Lemma 1, there exists a (u_0, u_t)-path P such that $V(P) = [u_{s+1}, u_t]$. Then $P\overrightarrow{C}[u_t, v_1]v_1u_s\overleftarrow{C}[u_s, v_0]R$ is an o-cycle containing all the vertices of $V(C) \cup V(R)$, a contradiction.

If $s = -1$, then by Claim 3, $v_1u_{-1} \notin \widetilde{E}(G)$.

If $s = 1$, then $\overleftarrow{C}[u_0, v_{-1}]v_{-1}v_1u_1\overrightarrow{C}[u_1, v_2]v_2v_0R$ is an o-cycle containing all the vertices of $V(C) \cup V(R)$, a contradiction.

If $s \in [2, \ell'+1]$, then $s-1 \in [1, \ell']$. By the definition of a composed graph, there exists $t \in [-k', -1]$ such that $G[u_t, u_{s-1}]$ is (u_t, u_0, u_{s-1})-composed. By Lemma 1, there exists a (u_0, u_t)-path P such that $V(P) = [u_t, u_{s-1}]$. Then $P\overleftarrow{C}[u_t, v_{-1}]v_{-1}v_1u_s\overrightarrow{C}[u_s, v_2]v_2v_0R$ is an o-cycle containing all the vertices of $V(C) \cup V(R)$, a contradiction. □

Let $G' = G[[u_{-k'-1}, u_{\ell'}] \cup \{v_0, v_1\}]$ and $G'' = G[[u_{-k'-1}, u_{\ell'+1}] \cup \{v_0, v_1\}]$. Then, similar to Claim 7.2, we obtain the following claim that we present without proof.

Claim 14.2. G'', and hence G', are $\{K_{1,3}, N_{1,1,2}\}$-free.

Similarly as for Claim 7, we can now complete the proof of Claim 14. □

Now we choose k', ℓ' such that

(1) $G[v_{-k'}, v_{\ell'}]$ is $(v_{-k'}, v_0, v_{\ell'})$-composed with canonical ordering $v_{-k'}, v_{-k'+1}, \ldots, v_{\ell'}$;

(2) any two nonadjacent vertices in $[v_{-k'}, v_{\ell'}]$ have degree sum less than n; and

(3) $k' + \ell'$ is as large as possible.

Similar to Claim 8, we obtain the following.

Claim 15. $(u_{-k'-1}, u_{\ell'})$ or $(u_{-k'}, u_{\ell'+1})$ or $(u_{-k'-1}, u_{\ell'+1})$ is u_0-good on C.

By Claim 13, $k' \leq r_2 - 2$ and $\ell' \leq r_1 - \ell - 2$.

From Claims 12 and 15, we conclude that there exists a cycle containing all vertices of $V(C) \cup V(R)$ by Lemma 3, a contradiction.

This completes the proof of Theorem 6.8.

6.4 Proof of Theorem 6.9

Let C be a longest cycle of G. Set $n = |V(G)|$ and $c = |V(C)|$, and assume that G is not hamiltonian, i.e., $c < n$. Then $V(G) \backslash V(C) \neq \emptyset$. Since G is 2-connected, there exists a (u_0, v_0)-path with length at least 2 which is internally-disjoint from C, where $u_0, v_0 \in V(C)$. Let $R = z_0 z_1 z_2 \cdots z_{r+1}$, where $z_0 = u_0$ and $z_{r+1} = v_0$, be such a path, and choose R as short as possible. Let r_1 and r_2 denote the number of interior vertices in the two subpaths of C from u_0 to v_0 (note that clearly $r_1 + r_2 + 2 = c$). We specify an orientation of C, and label the vertices of C using two distinct notations u_i and v_i, $-r_2 \leq i \leq r_1$, such that $\overrightarrow{C} = u_0 u_1 u_2 \cdots u_{r_1} v_0 u_{-r_2} u_{-r_2+1} \cdots u_{-1} u_0$ and $\overleftarrow{C} = v_0 v_1 v_2 \cdots v_{r_1} u_0 v_{-r_2} v_{-r_2+1} \cdots v_{-1} v_0$, where $u_\ell = v_{r_1+1-\ell}$ and $u_{-k} = v_{-r_2-1+k}$. Let H be the component of $G - C$ containing the vertices of $[z_1, z_r]$.

As in Section 6.3, we can prove the following claims.

Claim 1. Let $x \in V(H)$ and $y \in \{v_1, v_{-1}, u_1, u_{-1}\}$. Then $xy \notin \widetilde{E}(G)$.

Claim 2. $u_1 u_{-1} \in \widetilde{E}(G)$ and $v_1 v_{-1} \in \widetilde{E}(G)$.

Claim 3. $u_1 v_{-1} \notin \widetilde{E}(G)$, $u_{-1} v_1 \notin \widetilde{E}(G)$, $u_0 v_{\pm 1} \notin \widetilde{E}(G), u_{\pm 1} v_0 \notin \widetilde{E}(G)$.

Claim 4. Either $u_1 u_{-1}$ or $v_1 v_{-1}$ is in $E(G)$.

By Claim 4, without loss of generality, we assume that $u_1 u_{-1} \in E(G)$. Then $G[u_{-1}, u_1]$ is (u_{-1}, u_0, u_1)-composed.

Claim 5. If $G[u_{-k}, u_\ell]$ is (u_{-k}, u_0, u_ℓ)-composed, then $k \leq r_2 - 2$ and $\ell \leq r_1 - 2$.

The proof of Claim 5 is similar to that of Claim 6 in Section 6.3.

Now we prove the following claim.

Claim 6. If $G[u_{-k}, u_\ell]$ is (u_{-k}, u_0, u_ℓ)-composed with canonical ordering u_{-k}, $u_{-k+1}, \dots, u_\ell$, where $k \le r_2 - 2$ and $l \le r_1 - 2$, and any two nonadjacent vertices in $[u_{-k-1}, u_{\ell+1}]$ have degree sum less than n, then one of the following is true:

(1) $G[u_{-k-1}, u_\ell]$ is (u_{-k-1}, u_0, u_ℓ)-composed with canonical ordering u_{-k-1}, $u_{-k}, \dots, u_\ell$,

(2) $G[u_{-k}, u_{\ell+1}]$ is $(u_{-k}, u_0, u_{\ell+1})$-composed with canonical ordering u_{-k}, $u_{-k+1}, \dots, u_{\ell+1}$, or

(3) $G[u_{-k-1}, u_{\ell+1}]$ is $(u_{-k-1}, u_0, u_{\ell+1})$-composed with canonical ordering $u_{-k-1}, u_{-k}, \dots, u_{\ell+1}$.

Proof. Assume the opposite, which implies that for every vertex $u_s \in [u_{-k+1}, u_\ell]$, $u_{-k-1}u_s \notin E(G)$, and for every vertex $u_s \in [u_{-k}, u_{\ell-1}]$, $u_{\ell+1}u_s \notin E(G)$ and $u_{-k-1}u_{\ell+1} \notin E(G)$.

Claim 6.1. For every vertex $z \in \{z_1, z_2\}$ and $u_s \in [u_{-k-1}, u_{\ell+1}] \backslash \{u_0\}$, $zu_s \notin \widetilde{E}(G)$; and if $z_2u_0 \notin E(G)$, then also $z_2u_0 \notin \widetilde{E}(G)$.

This claim can be proved similarly as Claims 5 and 7.1 in Section 6.3.

Let $G' = G[[u_{-k-1}, u_\ell] \cup \{z_1, z_2\}]$ and $G'' = G[[u_{-k-1}, u_{\ell+1}] \cup \{z_1, z_2\}]$.

Similar to Claims 7.2 and 7.3 in Section 6.3, we can prove the following claims.

Claim 6.2. G'', and hence G', are $\{K_{1,3}, N_{1,1,2}, H_{1,1}\}$-free.

Claim 6.3. $N_{G'}(u_0) \backslash \{z_1, z_2\}$ is a clique.

Now, we define $N_i = \{x \in V(G') : d_{G'}(x, u_{-k-1}) = i\}$. Then $N_0 = \{u_{-k-1}\}$, $N_1 = \{u_{-k}\}$ and $N_2 = N_{G'}(u_{-k}) \backslash \{u_{-k-1}\}$.

By the definition of a composed graph, $|N_2| \ge 2$. If there are two vertices $x, x' \in N_2$ such that $xx' \notin E(G')$, then the graph induced by $\{u_{-k}, u_{-k-1}, x, x'\}$ is a claw. Thus N_2 is a clique.

We assume $u_0 \in N_j$, where $j \ge 2$. Then $z_1 \in N_{j+1}$; and $z_2 \in N_{j+1}$ if $z_2u_0 \in E(G)$ and $z_2 \in N_{j+2}$ if $z_2u_0 \notin E(G)$.

If $|N_i| = 1$ for some $i \in [2, j-1]$, say, $N_i = \{x\}$, then x is a cut vertex of the graph $G[u_{-k}, u_\ell]$. By the definition of a composed graph, $G[u_{-k}, u_\ell]$ is 2-connected. This implies $|N_i| \ge 2$ for every $i \in [2, j-1]$.

Claim 6.4. For $i \in [1, j]$, N_i is a clique.

Proof. If $i < j$, or $i = j$ and $z_2u_0 \notin E(G)$, then we can prove the assertion in a similar way as for Claim 7.4 in Section 6.3. Thus we assume that $i = j$ and $z_2u_0 \in E(G)$.

If $j = 2$, the assertion is true by the above analysis. So we assume that $j \geq 3$, and $N_{j-3}, N_{j-2}, N_{j-1}, N_{j+1}$ are nonempty, and $|N_{j-1}| \geq 2$.

First we prove that for every $x \in N_j \backslash \{u_0\}$, $u_0x \in E(G)$. We assume that $u_0x \notin E(G)$.

By Claim 6.1, $xz_1 \notin E(G)$. If u_0 and x have a common neighbor in N_{j-1}, denoted w, then let v be a neighbor of w in N_{j-2}; then the graph induced by $\{w, v, u_0, x\}$ is a claw, a contradiction. Thus u_0 and x have no common neighbors in N_{j-1}.

Let w be a neighbor of u_0 in N_{j-1}, and let w' be a neighbor of x in N_{j-1}. Then $u_0w', xw \notin E(G)$. Let v be a neighbor of w in N_{j-2}, and let u be a neighbor of v in N_{j-3}. If $w'v \notin E(G)$, then the graph induced by $\{w, v, w', u_0\}$ is a claw, a contradiction. Thus $w'v \in E(G)$, and then the graph induced by $\{v, u, w', x, w, u_0, z_1\}$ is an $N_{1,1,2}$, a contradiction.

Thus $u_0x \in E(G)$ for every $x \in N_j$. Then by Claim 6.3, N_j is a clique. □

If there exists some vertex $y \in N_{j+1}$ other than z_1 and z_2, then $yu_0 \notin E(G)$ by Claim 6.3. Let x be a neighbor of y in N_j, let w be a neighbor of u_0 in N_{j-1}, and let v be a neighbor of w in N_{j-2}. Then $xu_0 \in E(G)$ by Claim 6.4 and $xw \in E(G)$ by Claim 6.3. Then the graph induced by $\{w, v, x, y, u_0, z_1, z_2\}$ is an $N_{1,1,2}$ if $z_2u_0 \notin E(G)$, and is an $H_{1,1}$ if $z_2u_0 \in E(G)$, a contradiction. So we assume that all vertices in $[u_{-k}, u_\ell]$ are in $\bigcup_{i=1}^{j} N_i$.

If $u_\ell \in N_j$, then let w be a neighbor of u_0 in N_{j-1}, and let v be a neighbor of w in N_{j-2}. Then the graph induced by $\{w, v, u_0, z_1, u_\ell, u_{\ell+1}\}$ is an $N_{1,1,2}$ if $z_2u_0 \notin E(G)$, and is an $H_{1,1}$ if $z_2u_0 \in E(G)$, a contradiction. Thus we have that $u_\ell \notin N_j$ and then $j \geq 3$.

Let $u_\ell \in N_i$, where $i \in [2, j-1]$. If u_ℓ has a neighbor in N_{i+1}, then let y be a neighbor of u_ℓ in N_{i+1}, and let w be a neighbor of u_ℓ in N_{i-1}. Then the graph induced by $\{u_\ell, w, y, u_{\ell+1}\}$ is a claw, a contradiction. Thus u_ℓ has no neighbors in N_{i+1}.

Let $x \in N_i$ be a vertex other than u_ℓ that has a neighbor y in N_{i+1} such that it has a neighbor z in N_{i+2}. Let w be a neighbor of x in N_{i-1}, and let v be a neighbor of w in N_{i-2}. If $u_\ell w \notin E(G)$, then the graph induced by $\{x, w, u_\ell, y\}$ is a claw, a contradiction. Thus $u_\ell w \in E(G)$. Then the graph induced by $\{w, v, u_\ell, u_{\ell+1}, x, y, z\}$ is an $N_{1,1,2}$, a contradiction.

This completes the proof of Claim 6. □

Now we choose k, ℓ such that

(1) $G[u_{-k}, u_\ell]$ is (u_{-k}, u_0, u_ℓ)-composed with canonical ordering $u_{-k}, u_{-k+1}, \ldots, u_\ell$;

(2) any two nonadjacent vertices in $[u_{-k}, u_\ell]$ have degree sum less than n; and

(3) $k + \ell$ is as large as possible.

Similar to Claims 8 and 9 in Section 6.3, we obtain the following claims.

Claim 7. (u_{-k-1}, u_ℓ) or $(u_{-k}, u_{\ell+1})$ or $(u_{-k-1}, u_{\ell+1})$ is u_0-good on C.

Claim 8. There exist $v_{-k'} \in V(\overrightarrow{C}[v_{-1}, u_{-k-1}])$ and $v_{\ell'} \in V(\overleftarrow{C}[v_1, u_{\ell+1}])$ such that $(v_{-k'}, v_{\ell'})$ is v_0-good on C.

From Claims 7 and 8, we conclude that there exists a cycle containing all the vertices of $V(C) \cup V(R)$ by Lemma 3, a contradiction.

This completes the proof of Theorem 6.9.

Chapter 7

Forbidden pairs for homogeneously traceable graphs

7.1 Introduction

We call a graph G *hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle containing all its vertices, *traceable* if it contains a *Hamilton path*, i.e., a path containing all its vertices, and *Hamilton-connected* if for every pair of vertices x, y of G, G contains a Hamilton path starting from x and terminating in y. We say that G is *homogeneously traceable* if for every vertex x of G, G contains a Hamilton path starting from x. Homogeneously traceable graphs have been introduced by Skupień (see, e.g., [34]), but we do not know whether he is the original source of the concept.

Note that a Hamilton-connected graph (on at least three vertices) is hamiltonian, that a hamiltonian graph is homogeneously traceable, and that a homogeneously traceable graph is traceable, but that the reverse statements do not hold in general.

If a graph is connected and P_3-free, then it is a *complete graph*, i.e., its vertex set is a *clique*, i.e., all its vertices are mutually adjacent, and hence it is (homogeneously) traceable, and hamiltonian if it has order at least 3. In fact, it is not hard to show that the statement 'every connected H-free graph is traceable' only holds if $H = P_3$. The case with pairs of forbidden subgraphs (different from P_3) is much more interesting. For a connected graph to be traceable or hamiltonian, the following theorem is one of the earliest of this kind.

Theorem 7.1 (Duffus, Gould and Jacobson [21]). *Let G be a $\{K_{1,3}, N\}$-free graph.*

(1) *If G is connected, then G is traceable.*
(2) *If G is 2-connected, then G is hamiltonian.*

Obviously, if H is an induced subgraph of N, then $\{K_{1,3}, H\}$-free instead of $\{K_{1,3}, N\}$-free yields the same conclusions in the above theorem. A natural problem that, as far as we know, was considered for the first time in the PhD Thesis of Bedrossian [3], is to characterize all pairs of forbidden subgraphs for hamiltonicity (and other graph properties). Faudree and Gould [24] later refined this approach by adding a lower bound on the number of vertices of the graph G in order to avoid small, more or less pathological, cases. Restricting our attention to traceability, they proved that (apart from trivial cases) the claw and any of the induced subgraphs of the net are the only forbidden pairs for the property of being traceable.

Theorem 7.2 (Faudree and Gould [24]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is P_4, C_3, Z_1, B or N.*

In the same paper, they discuss analogous results for other hamiltonian properties. For many of these properties counterparts of Theorem 7.2 have been established, but for Hamilton-connectedness only partial results are known to date. We refer to [24] for more details. The property of being homogeneously traceable was not addressed in [24] and, as far as we are aware, has not been considered before. Recently, similar questions related to the existence of perfect matchings and 2-factors have been studied. We refer the interested reader to [27,31] and [2,23,28], respectively, for more details.

In this chapter we solve the analogous problem for homogeneously traceable graphs, so we are going to characterize the pairs of connected forbidden induced subgraphs that imply that a given graph is homogeneously traceable.

Note that if a graph contains a cut vertex v, it cannot be homogeneously traceable since there exists no Hamilton path starting at v. So, apart from K_1 and K_2, all homogeneously traceable graphs are 2-connected. Thus we only consider 2-connected graphs. As noted before, if a connected graph G is P_3-free, then it is a complete graph, and hence trivially homogeneously traceable, and in fact it is easy to prove the following statement. We postpone the proof of the 'only-if' part of the next statement to Section 7.2.

Theorem 7.3. *The only connected graph S such that every 2-connected S-free graph is homogeneously traceable is P_3.*

A natural and more interesting problem is to consider pairs of forbidden subgraphs for this property. In this chapter, we characterize all such pairs by proving the following result. We refer to Figure 7.1 for an illustration of the relevant graphs.

Theorem 7.4. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is homogeneously traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $B_{1,4}$, $B_{2,3}$ or $N_{1,1,3}$.*

In Section 7.2, we prove the 'only-if' part of the statements of Theorems 7.3 and 7.4, while the 'if' part of the statement of Theorem 7.4 is deduced from the following three theorems that will be proved in Sections 7.5, 7.6 and 7.7, respectively.

Let G be a 2-connected graph.

Theorem 7.5. *If G is $\{K_{1,3}, B_{1,4}\}$-free, then G is homogeneously traceable.*

Theorem 7.6. *If G is $\{K_{1,3}, B_{2,3}\}$-free, then G is homogeneously traceable.*

Theorem 7.7. *If G is $\{K_{1,3}, N_{1,1,3}\}$-free, then G is homogeneously traceable.*

Section 7.4 contains the common set-up for the proofs of the above three theorems and some common preliminary observations. We present some general observations on claw-free graphs in Section 7.3.

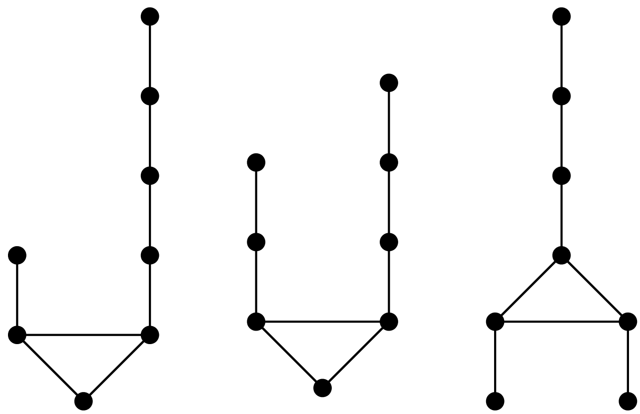


Figure 7.1: The graphs $B_{1,4}$, $B_{2,3}$ and $N_{1,1,3}$

7.2 The 'only-if' part of the proofs of Theorems 7.3 and 7.4

We first sketch some families of graphs that are not homogeneously traceable (see Figure 7.2). In each of the graphs in Fig. 7.2, we indicated one of the vertices by a double circle; it is easy to check that this vertex cannot be the starting vertex of a Hamilton path. When we say that a graph is of *type* G_i we mean that it is one particular, but arbitrarily chosen member of the family indicated by G_i in Figure 7.2.

If S is a connected graph such that every 2-connected S-free graph is homogeneously traceable, then S must be a common induced subgraph of all graphs of type G_1, G_2 and G_3. Note that the largest common induced connected subgraph of graphs of type G_1, G_2 and G_3 is a P_3, so we have that $S = P_3$. This completes the proof of the 'only-if' part of the statement of Theorem 7.3.

Let R and S be two connected graphs other than P_3 such that every 2-connected $\{R, S\}$-free graph is homogeneously traceable. Then R or S must

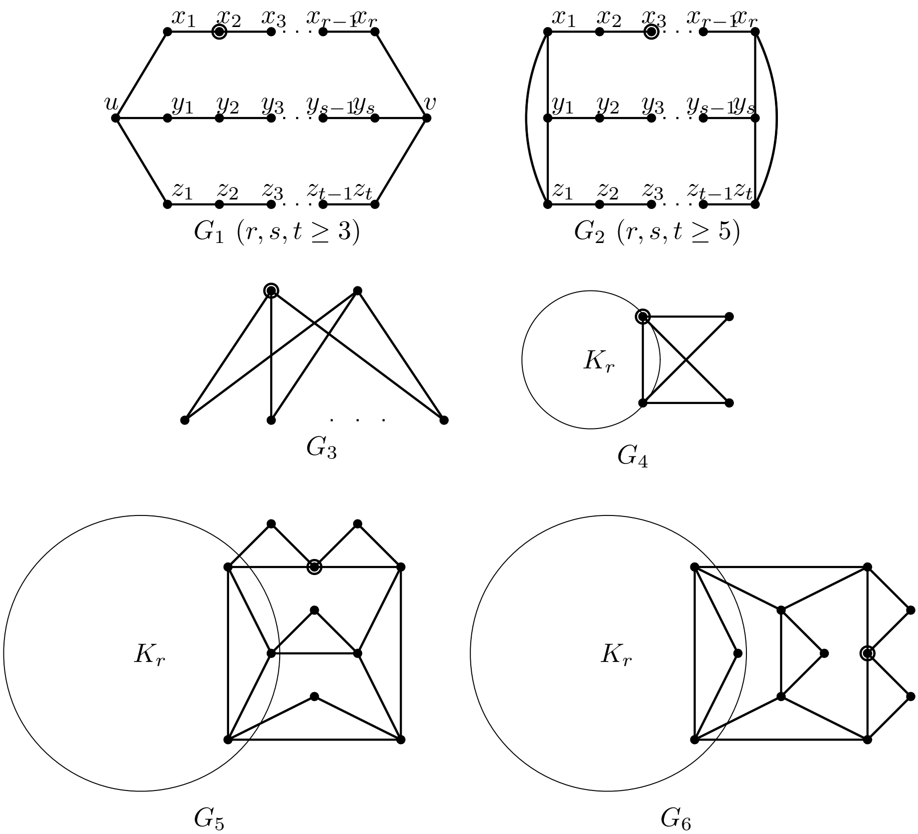


Figure 7.2: Some graphs that are not homogeneously traceable (I)

be an induced subgraph of all graphs of type G_1. Without loss of generality, we assume that R is an induced subgraph of a graph of type G_1. If $R \neq K_{1,3}$, then R must contain an induced P_4. Note that the graphs of type G_3 and G_4 are all P_4-free, so they must contain S as an induced subgraph. Since the only common induced connected subgraph of the graphs of type G_3 and G_4 other than P_3 is a $K_{1,3}$, we have that $S = K_{1,3}$. This implies that R or S must be a $K_{1,3}$.

Let $R = K_{1,3}$. Note that the graphs of type G_2 are claw-free, so S must be an induced connected subgraph of all graphs of type G_2. The common induced connected subgraphs of such graphs have the form P_i, Z_i, $B_{i,j}$ or $N_{i,j,k}$. Note that graphs of type G_5 are claw-free and do not contain an induced P_8, Z_5 or $N_{1,1,4}$, and that graphs of type G_6 are claw-free and do not contain an induced $N_{1,2,2}$. So R must be an induced connected subgraph of P_7, Z_4, $B_{1,4}$, $B_{2,3}$ or $N_{1,1,3}$. Since P_7 and Z_4 are induced subgraphs of $B_{1,4}$, R must be an induced connected subgraph of $B_{1,4}$, $B_{2,3}$ or $N_{1,1,3}$. This completes the proof of the 'only-if' part of the statement of Theorem 7.4.

7.3 Some preliminaries

Let G be a graph. For a subgraph H of G, when no confusion can arise we also use H to denote the vertex set of H; and similarly, for a subset S of $V(G)$, we also use S to denote the subgraph of G induced by S. For two vertices u and v of G, we use $d_H(u, v)$ to denote the *distance* between u and v in H, i.e., the length of a shortest path between u and v with all edges in H.

We first prove some easy but useful observations on claw-free graphs.

Lemma 1. *Let G be a 2-connected claw-free graph and let $\{x, y\}$ be a vertex cut of G. Then the following statements hold:*
(1) *$G - \{x, y\}$ has exactly two components; and*
(2) *if x_1 and x_2 are two neighbors of x in the same component of $G - \{x, y\}$, then $x_1x_2 \in E(G)$.*

Proof. Note that each component H of $G - \{x, y\}$ contains a neighbor of x; otherwise y is a cut vertex of G, a contradiction.

If there are at least three components of $G - \{x, y\}$, then let H_1, H_2 and H_3 be three such components. Let x_1, x_2 and x_3 be neighbors of x in H_1, H_2

and H_3, respectively. Then the subgraph induced by $\{x, x_1, x_2, x_3\}$ is a claw, a contradiction. Thus we conclude that $G - \{x, y\}$ has exactly two components.

Let x_1 and x_2 be two neighbors of x in the same component of $G - \{x, y\}$. If $x_1x_2 \notin E(G)$, then let x' be a neighbor of x in the other component of $G - \{x, y\}$. Then the subgraph induced by $\{x, x_1, x_2, x'\}$ is a claw, a contradiction. Thus we have $x_1x_2 \in E(G)$. □

Throughout the remainder of this chapter, by the word *cut* we will always refer to a vertex cut with exactly two vertices.

We say that two disjoint subsets or subgraphs S and T of G are *joined* if at least one vertex of S is adjacent to a vertex of T in G.

Let B and C be two subgraphs of G (possibly not disjoint), and let H be a subgraph of G that is disjoint from B and C. If P is a path with one end vertex x in B, one end vertex y in C, and its internal vertex set $V(P) \setminus \{x, y\} = V(H)$, then we call P *a perfect path of H to B and C (in G)*; if $B = C$, then we call P *a perfect path of H to B (in G)*. If there is a perfect path of H to B (and C), then we say that H supports a perfect path to B (and C).

We will frequently use the following argumentation in the next sections. Let H be a 2-connected claw-free subgraph of G, and let r, s be a pair of distinct vertices of H. Then $H - s$ is a connected graph. We consider the neighborhood structure of r in $H - s$ by defining, for integers $i = 0, 1, \ldots,$

$$N_i(r) = \{v \in V(H - s) : d_{H-s}(v, r) = i\} \text{ and } j = \max\{i : N_i(r) \neq \emptyset\}.$$

For a vertex $v \in N_i(r)$, the index i is referred to as the *level* of v. If these neighborhoods are complete or 'nearly' complete, we can deduce the existence of a Hamilton path of H between r and s, as follows.

Lemma 2. *Let H be a 2-connected claw-free graph, let r and s be a pair of distinct vertices of H, and let $N_i(r)$ and j be as defined above. Suppose there is an integer j' with $1 \leq j' \leq j$, such that*
(1) for every i with $1 \leq i \leq j'$, $N_i(r)$ is a clique;
(2) $N(s) \setminus \{r\}$ is a clique; and
(3) $j' = j$, or for every component C of $\bigcup_{i=j'+1}^{j} N_i(r)$: if s is not adjacent to a vertex of C, then C supports a perfect path to $N_{j'}(r)$; if s is adjacent to a vertex of C, then C supports a perfect path to $N_{j'}(r)$ and s.

Then there is a Hamilton path of H between r and s.

Proof. For convenience we let N_i denote $N_i(r)$ throughout this proof.

If $j' \leq j-1$, then let $\mathcal{H} = \{H_1, H_2 \ldots, H_k\}$ be the set of components of $\bigcup_{i=j'+1}^{j} N_i$. For every i with $1 \leq i \leq k$, if s is not adjacent to a vertex of H_i, then let R_i be a perfect path of H_i to $N_{j'}$, and let y_i, y_i' be the two end vertices of R_i; if s is adjacent to a vertex of H_i, then let R_i be a perfect path of H_i to $N_{j'}$ and s, and let y_i be the end vertex of R_i other than s.

If two components H_i and $H_{i'}$ have a common neighbor y in $N_{j'}$, then let z be a neighbor of y in H_i, let z' be a neighbor of y in $H_{i'}$, and let x be a neighbor of y in $N_{j'-1}$. Then the subgraph induced by $\{y, x, z, z'\}$ is a claw, a contradiction. This implies that any two perfect paths R_i and $R_{i'}$ have no common end vertices in $N_{j'}$; since $N(s) \setminus \{r\}$ is a clique, R_i and $R_{i'}$ cannot have s as a common end vertex either.

Note that $N_0 = \{r\}$. Let $s' \in N_{j''} \setminus \{r\}$ be a neighbor of s such that its level j'' is as large as possible, where $1 \leq j'' \leq j$ (such a vertex exists since H is 2-connected).

We prove the following five claims in order to show that there is a Hamilton path of H between r and s.

Claim 1. If $j'' \leq j'-1$, then $\bigcup_{i=j'}^{j} N_i$ contains a perfect path to $N_{j'-1}$.

Proof. We first assume that $j' = j$. If N_j has only one vertex x, then by the 2-connectedness of H, x has at least two neighbors in N_{j-1}. Let w, w' be two neighbors of x in N_{j-1}. Then $R = wxw'$ is a perfect path of N_j to N_{j-1}.

If N_j has at least two vertices, then by the 2-connectedness of H, N_j is joined to N_{j-1} by (at least) two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in N_j$ and $w, w' \in N_{j-1}$. Let R' be a Hamilton path of (the clique) N_j from x to x'. Then $R = wxR'x'w'$ is a perfect path of N_j to N_{j-1}.

Thus we assume that $j' \leq j-1$. By the 2-connectedness of H, $N_{j'}$ is joined to $N_{j'-1}$ by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in N_{j'}$ and $w, w' \in N_{j'-1}$.

We first assume that one vertex of x and x' is not an end vertex of some perfect path. Without loss of generality, we assume that x is not an end vertex of some perfect path. If x' is also not an end vertex of some perfect path, then let T be a path of $N_{j'}$ from x to y_1 passing through all the vertices in $N_{j'} \setminus \bigcup_{i=1}^{k}\{y_i, y_i'\} \setminus \{x'\}$. Then $R = wxTy_1R_1y_1' \cdots y_kR_ky_k'x'w'$ is a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$.

If x' is an end vertex of some perfect path, then without loss of generality, we assume that $x' = y_k'$. Let T be a path of $N_{j'}$ from x to y_1 passing through all the vertices in $N_{j'} \setminus \bigcup_{i=1}^{k}\{y_i, y_i'\}$. Then $R = wxTy_1R_1y_1' \cdots y_kR_ky_k'w'$ is a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$.

Suppose now that both x and x' are end vertices of some perfect paths. If there is a vertex x'' in $N_{j'}$ other than $\bigcup_{i=1}^{k}\{y_i, y_i'\}$, then let w'' be a neighbor of x'' in $N_{j'-1}$. Without loss of generality, we assume that $w'' \neq w$. Then xw and $x''w''$ are two independent edges joining $N_{j'}$ to $N_{j'-1}$ such that x'' is not an end vertex of some perfect path. By the previous arguments, we can find a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$. So we assume that there are no vertices in $N_{j'}$ other than $\bigcup_{i=1}^{k}\{y_i, y_i'\}$.

If x and x' are end vertices of two distinct perfect paths, then without loss of generality, we assume that $x = y_1$ and $x' = y_k'$. Then $R = wy_1R_1y_1' \cdots y_kR_ky_k'w'$ is a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$.

Suppose now that x and x' are the two end vertices of a common perfect path. If there is a second perfect path, then let x'' be an end vertex of a second perfect path and w'' be a neighbor of x'' in $N_{j'-1}$. Without loss of generality, we assume that $w'' \neq w$. Then xw and $x''w''$ are two independent edges joining $N_{j'}$ to $N_{j'-1}$ such that x and x'' are end vertices of two distinct perfect paths. By the previous arguments, we can find a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$.

So finally we assume that there is only one perfect path R_1. Without loss of generality, we assume that $x = y_1$ and $x' = y_1'$. Then $R = wy_1R_1y_1'w'$ is a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$. □

Claim 2. If $j'' \leq j' - 1$, then for every i with $j'' + 1 \leq i \leq j'$, $\bigcup_{i'=i}^{j} N_{i'}$ supports a perfect path to N_{i-1}.

Proof. We prove the claim by induction on $j' - i$.

If $i = j'$, then by Claim 1, $\bigcup_{i'=j'}^{j} N_{i'}$ supports a perfect path to $N_{j'-1}$. Thus we assume that $j'' + 1 \leq i \leq j' - 1$.

By the induction hypothesis, there is a perfect path R' of $\bigcup_{i'=i+1}^{j} N_{i'}$ to N_i. Let y and y' be the two end vertices of R'.

By the 2-connectedness of H, N_i is joined to N_{i-1} by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in N_i$ and $w, w' \in N_{i-1}$.

We first assume that x, x' and y, y' are two distinct pairs. Without loss of generality, we assume that $x \neq y, y'$. If $x' \neq y, y'$, then let T be a path of N_i from x to y passing through all the vertices in $N_i \setminus \{x', y'\}$. Then $R = wxTyR'y'x'w'$ is a perfect path of $\bigcup_{i'=i}^{j} N_{i'}$ to N_{i-1}; if $x' = y$ or y', then without loss of generality, we assume that $x' = y'$. Let T be a path of N_i from x to y passing through all the vertices in $N_i \setminus \{x'\}$. Then $R = wxTyR'x'w'$ is a perfect path of $\bigcup_{i'=i}^{j} N_{i'}$ to N_{i-1}.

Suppose now that x, x' and y, y' are the same pair.

If there is a third vertex x'' in N_i other that x and x', then let w'' be a neighbor of x'' in N_{i-1}. Without loss of generality, we assume that $w'' \neq w$. Then xw and $x''w''$ are two independent edges joining N_i to N_{i-1} such that x, x'' and y, y' are two distinct pairs. By the previous arguments, we can find a perfect path of $\bigcup_{i'=i}^{j} N_{i'}$ to N_{i-1}.

Finally we assume that there are only the two vertices x and x' in N_i. Then $R = wxR'x'w'$ is a perfect path of $\bigcup_{i'=i}^{j} N_{i'}$ to N_{i-1}. □

Claim 3. If $j'' \leq j' - 1$, then $\bigcup_{i=j''}^{j} N_i$ contains a perfect path to $N_{j''-1}$ and s.

Proof. By Claim 2, there is a perfect path R' of $\bigcup_{i=j''+1}^{j} N_i$ to $N_{j''}$. Let y and y' be the two end vertices of R'.

We first assume that there is a vertex x in $N_{j''}$ other than y, y' and s'. Let w be a neighbor of x in $N_{j''-1}$. If $s' \neq y, y'$, then let T be a path of $N_{j''}$ from x to y passing through all the vertices in $N_{j''} \setminus \{y', s'\}$. Then $R = wxTyR'y's's$ is a perfect path of $\bigcup_{i=j''}^{j} N_i$ to $N_{j''-1}$ and s; if $s' = y$ or y', then without loss of generality, we assume that $s' = y'$. Let T be a path of $N_{j''}$ from x to y passing through all the vertices in $N_{j''} \setminus \{y'\}$. Then $R = wxTyR'y's$ is a perfect path of $\bigcup_{i=j''}^{j} N_i$ to $N_{j''-1}$ and s.

Suppose now that there are no vertices in $N_{j''}$ other than y, y' and s'. If $s' \neq y, y'$, then let w be a neighbor of y in $N_{j''-1}$. Then $R = wyR'y's's$ is a perfect path of $\bigcup_{i=j''}^{j} N_i$ to $N_{j''-1}$ and s; if $s' = y$ or y', then without loss of generality, we assume that $s' = y'$. Let w be a neighbor of y in $N_{j''-1}$. Then $R = wyR'y's$ is a perfect path of $\bigcup_{i=j''}^{j} N_i$ to $N_{j''-1}$ and s. □

Claim 4. If $j'' \geq j'$, then $\bigcup_{i=j'}^{j} N_i$ supports a perfect path to $N_{j'-1}$ and s.

Proof. We first assume that $j' = j$, and thus $j'' = j$. If N_j consists of the vertex s', then let w be a neighbor of s' in N_{j-1}. Then $R = ws's$ is a perfect path of N_j to N_{j-1} and s; if N_j contains at least two vertices, then let x be a vertex in N_j other than s', let w be a neighbor of x in N_{j-1}, and let R' be a Hamilton path of N_j from x to s'. Then $R = wxR's's$ is a perfect path of N_j to N_{j-1} and s.

Next we assume that $j' \leq j-1$.

First we assume that s is not adjacent to any vertex in $\mathcal{H}$. Then s' is a neighbor of s in $N_{j'}$.

We first treat the case that s' is not an end vertex of some perfect path. If there is a vertex x in $N_{j'}$ other than $\bigcup_{i=1}^{k}\{y_i, y_i'\} \cup \{s'\}$, then let w be a neighbor of x in $N_{j'-1}$, and let T be a path of $N_{j'}$ from x to y_1 passing through all the vertices in $N_{j'} \setminus \bigcup_{i=1}^{k}\{y_i, y_i'\} \setminus \{s'\}$. Then $R = wxTy_1R_1y_1' \cdots y_kR_ky_k's's$ is a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$ and s; if there are no vertices in $N_{j'}$ other than $\bigcup_{i=1}^{k}\{y_i, y_i'\} \cup \{s'\}$, then let w be a neighbor of y_1 in $N_{j'-1}$. Then $R = wy_1R_1y_1' \cdots y_kR_ky_k's's$ is a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$ and s.

Next we treat the case that s' is an end vertex of some perfect path. Without loss of generality, we assume that $s' = y_k'$. If there is a vertex x in $N_{j'}$ other than $\bigcup_{i=1}^{k}\{y_i, y_i'\}$, then let w be a neighbor of x in $N_{j'-1}$, and let T be a path of $N_{j'}$ from x to y_1 passing through all the vertices in $N_{j'} \setminus \bigcup_{i=1}^{k}\{y_i, y_i'\}$. Then $R = wxTy_1R_1y_1' \cdots y_kR_ky_k's$ is a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$ and s; if there are no vertices in $N_{j'}$ other than $\bigcup_{i=1}^{k}\{y_i, y_i'\}$, then let w be a neighbor of y_1 in $N_{j'-1}$. Then $R = wy_1R_1y_1' \cdots y_kR_ky_k's$ is a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$ and s.

Suppose now that s is adjacent to a vertex of some component of $\mathcal{H}$. Note that $N(s) \setminus \{r\}$ is a clique and that s is adjacent to at most one component of $\mathcal{H}$. Without loss of generality, we assume that s is adjacent to a vertex of H_k, and thus s is the end vertex of R_k other than y_k. If there is a vertex x in $N_{j'}$ other than $\bigcup_{i=1}^{k-1}\{y_i, y_i'\} \cup \{y_k\}$, then let w be a neighbor of x in $N_{j'-1}$, and let T be a path of $N_{j'}$ from x to y_1 passing through all the vertices in $N_{j'} \setminus \bigcup_{i=1}^{k-1}\{y_i, y_i'\} \setminus \{y_k\}$. Then $R = wxTy_1R_1y_1' \cdots y_kR_k$ is a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$ and s; if there are no vertices in $N_{j'}$ other than $\bigcup_{i=1}^{k-1}\{y_i, y_i'\} \cup \{y_k\}$, then let w be a neighbor of y_1 in $N_{j'-1}$. Then $R = wy_1R_1y_1' \cdots y_kR_k$ is a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$ and s. □

Claim 5. For every i with $1 \leq i \leq \min\{j', j''\}$, $\bigcup_{i'=i}^{j} N_{i'}$ supports a perfect path to N_{i-1} and s.

Proof. We prove the claim by induction on $\min\{j', j''\} - i$.

If $i = \min\{j', j''\}$, then by Claims 3 and 4, $\bigcup_{i'=i}^{j} N_{i'}$ supports a perfect path to N_{i-1} and s. Thus we assume that $1 \le i \le \min\{j', j''\} - 1$.

By the induction hypothesis, there is a perfect path R' of $\bigcup_{i'=i+1}^{j} N_{i'}$ to N_i and s. Let y be the end vertex of R' other than s.

If there is a second vertex x in N_i other than y, then let w be a neighbor of x in N_{i-1}, and let T be a Hamilton path of N_i from x to y. Then $R = wxTyR'$ is a perfect path of $\bigcup_{i'=i}^{j} N_{i'}$ to N_{i-1} and s.

Thus we assume that N_i consists of the vertex y. Let w be a neighbor of y in N_{i-1}. Then $R = wyR'$ is a perfect path of $\bigcup_{i'=i}^{j} N_{i'}$ to N_{i-1} and s. □

Taking $i = 1$ in Claim 5, we conclude that there exists a Hamilton path of H from r to s. This completes the proof of Lemma 2. □

7.4 A common set-up for the proofs

The three proofs of Theorems 7.5–7.7 are modeled along the same lines and use the same case distinctions. To avoid too much repetition of the arguments we give the generic set-up for all three proofs and treat some of the subcases simultaneously in this section.

Let G be a 2-connected $\{K_{1,3}, F\}$-free graph, where $F = B_{1,4}$, $B_{2,3}$ or $N_{1,1,3}$. We are going to prove that G is homogeneously traceable by induction on $|V(G)|$. If $|V(G)| = 3$, the result is trivially true. So we assume that $|V(G)| \ge 4$ and that the statement holds for any 2-connected $\{K_{1,3}, F\}$-free graph with order $n < |V(G)|$.

Let v be an arbitrary vertex of G. It is sufficient to prove that G contains a Hamilton path starting from r.

If $G - r$ is 2-connected, then we consider a neighbor r' of r in G. By the induction hypothesis, $G - r$ contains a Hamilton path P starting from r'. Then $rr'P$ is a Hamilton path of G starting from r, and the statement holds.

So we assume that $G - r$ is *separable*, i.e., has a cut vertex. We consider the *blocks* of $G - r$, i.e., the maximal subgraphs of $G - r$ that do not have a cut vertex, so these blocks are either isomorphic to K_2 or 2-connected. We say that a block is *trivial* if it is isomorphic to K_2. An *end block* is a block containing exactly one cut vertex of $G - r$; the other blocks are called *inner blocks*. Except for the cut vertex, all other vertices of an end block are called *inner vertices*.

Note that every end block of $G - r$ contains an inner vertex adjacent to r, and that $G - r$ has at least two end blocks. Since G is claw-free, we deduce that there are exactly two end blocks of $G - r$. This implies that the $p + 1 \geq 2$ blocks of $G - r$ can be denoted as $B_0, B_1, B_2, \ldots, B_p$ with cut vertices s_i, $1 \leq i \leq p$, of $G - r$ common to B_{i-1} and B_i, and s_0 and s_{p+1} two neighbors of r contained in $B_0 - s_1$ and $B_p - s_p$, respectively.

We distinguish two main cases: there is a nontrivial inner block or all inner blocks are trivial. In the former case we need basically separate approaches except if we assume another nontrivial block. We complete this section by first treating the common subcase that there is a nontrivial inner block and another nontrivial block. We also give some generic observations for the other subcases and treat the subcase that all inner blocks are trivial simultaneously. The other subcases are treated in detail separately in Sections 7.5-7.7.

The case with a nontrivial inner block and another nontrivial block

Suppose B_q is a nontrivial inner block, where $1 \leq q \leq p - 1$. Here we deal with the subcase that there is another nontrivial block B_r (either inner or end block). In this case, we only need the induction hypothesis. Let Q_q be a shortest path in B_q from s_q to s_{q+1}, and Q_r a shortest path in B_r from s_r to s_{r+1}. Since B_q (B_r) is nontrivial and 2-connected, Q_q (Q_r) must miss some vertices in B_q (B_r). Let G_q be the subgraph induced by $V(G - B_q) \cup V(Q_q)$, and let G_r be the subgraph induced by $V(G - B_r) \cup V(Q_r)$. By the induction hypothesis, G_r contains a Hamilton path H_r starting from r. Clearly s_q and s_{q+1} are two cut vertices of $G_r - r$, so the subpath Q'_q of H_r from s_q to s_{q+1} is a Hamilton path of B_q. Similarly, G_q contains a Hamilton path H_q starting from r, and Q_q is the subpath of H_q from s_q to s_{q+1}. Let P be the path obtained from H_q by replacing Q_q by Q'_q. Then P is a Hamilton path of G starting from r, and the statement holds.

This completes the proof for Theorems 7.5-7.7 in case $G - r$ contains a nontrivial inner block and another nontrivial (inner or end) block.

The case with one nontrivial inner block and all other blocks trivial

Next we assume that all the blocks of $G - r$ other than B_q are trivial. Then the structure of the blocks implies that it is sufficient to show that there exists a Hamilton path in B_q between s_q and s_{q+1}. The subcases can be treated by first analyzing the structure of the neighborhoods of s_q in $B_q - s_{q+1}$ and then using Lemma 2.

Set

$$N_i = \{u \in B_q - s_{q+1} : d_{B_q - s_{q+1}}(u, s_q) = i\}, \text{ and } j = \max\{i : N_i \neq \emptyset\}.$$

Note that $N_0 = \{s_q\}$ and $N_1 = N_{B_q}(s_q) \setminus \{s_{q+1}\}$.

Recall that B_q is nontrivial, hence it is 2-connected. First we prove the following easy common observation.

Observation 1. *$N_{B_q}(s_q)$ is a clique and $N_{B_q}(s_{q+1})$ is a clique.*

Proof. If there are two neighbors x and x' of s_q in B_q such that $xx' \notin E(G)$, then the subgraph induced by $\{s_q, s_{q-1}, x, x'\}$ is a claw, a contradiction. Similarly we can prove that $N_{B_q}(s_{q+1})$ is a clique. □

Note that Lemma 3 implies that N_1 is a clique. To analyze the structure of the other N_i we use slightly different arguments depending on the forbidden subgraph F. Although there is a lot of commonality, in Sections 7.5-7.7 we use the above set-up and notation, and treat the subcase that the inner block B_q is nontrivial and all other blocks are trivial separately for Theorems 7.5-7.7.

In the three different proofs for this subcase, we will implicitly prove the following technical lemma. We state it here already because we want to apply it in the next subcase as well. It will be clear from Sections 7.5-7.7 that the proof of this lemma is different for the different choices of the forbidden subgraph F, and that it would have been a bad idea to include the proof at this point.

Lemma 3. *Let G be a 2-connected $\{K_{1,3}, F\}$-free graph, where $F = B_{1,4}$, $B_{2,3}$ or $N_{1,1,3}$. Let H be an induced 2-connected subgraph of G, and let r, s be a pair of distinct vertices of H. Suppose:*

(1) $N_H(r)$ is a clique;

(2) $N_H(s) \setminus \{r\}$ is a clique;

(3) there is an induced path P in G of length at least 3 with origin r, with $V(P) \cap V(H) = \{r\}$, and such that in G there are no edges joining $V(H) \setminus \{s\}$ and $V(P)$ except the first edge of P;

(4) if the distance between r and s in H is at least 4, there is a neighbor of r outside H that is nonadjacent to $V(H) \setminus \{r\}$.

Then H has a Hamilton path between r and s.

The case that all inner blocks are trivial

In the final case we assume that all inner blocks of $G - r$ are trivial. If $p \geq 2$, we let Q be the (unique) path from s_1 to s_p with all internal vertices outside $B_0 \cup B_p$; if $p = 1$, we let Q consist of s_1. We recall that B_0 is either trivial or 2-connected. Using the induction hypothesis in the latter case, this implies that there is a Hamilton path in B_0 starting from s_1. Similarly, there is a Hamilton path in B_p starting from s_p. If there exists a Hamilton path in $B_0 \cup \{r\}$ from r to s_1, then combining it with Q (if $p \geq 2$) and the Hamilton path in B_p starting from s_p, we obtain a Hamilton path in G starting from r. By symmetry, it is sufficient to prove the claim that there is a Hamilton path in $B_0 \cup \{r\}$ from r to s_1 or a Hamilton path in $B_p \cup \{r\}$ from r to s_p.

If B_0 or B_p is trivial, then the claim clearly holds. So we assume that neither B_0 nor B_p is trivial.

If r has only one neighbor s_0 in B_0, then let $B_0' = B_0$ and $r_0 = s_0$; otherwise let B_0' be the subgraph induced by $B_0 \cup \{r\}$ and let $r_0 = r$. Analogously, if r has only one neighbor s_{p+1} in B_p, then let $B_p' = B_p$ and $r_{p+1} = s_{p+1}$; otherwise let B_p' be the subgraph induced by $B_p \cup \{r\}$ and let $r_{p+1} = r$. Now it is sufficient to prove that B_0' contains a Hamilton path from r_0 to s_1, or B_p' contains a Hamilton path from r_{p+1} to s_p.

By our choice of B_0' and B_p', we have that B_0' and B_p' are both 2-connected. Moreover, we can prove the following two observations by only using the claw-freeness of G.

Observation 2. *$N_{B_0'}(r_0)\setminus\{s_1\}$, $N_{B_0'}(s_1)\setminus\{r_0\}$, $N_{B_p'}(s_p)\setminus\{r_{p+1}\}$ and $N_{B_p'}(r_{p+1}) \setminus\{s_p\}$ are all cliques.*

Proof. Suppose that $N_{B_0'}(r_0) \setminus \{s_1\}$ is not a clique. Let x, x' be two neighbors of r_0 in $B_0' - s_1$ that are nonadjacent. If $r_0 = v$, then the subgraph induced by $\{v, s_{p+1}, x, x'\}$ is a claw, a contradiction. If $r_0 = s_0$, then the subgraph induced by $\{s_0, v, x, x'\}$ is a claw, a contradiction.

The other assertions can be proved in a similar way. □

Observation 3. *$N_{B_0'}(r_0)$ or $N_{B_p'}(r_{p+1})$ is a clique. Moreover, if $r_0 s_1 \notin E(G)$ or $r_{p+1} s_p \notin E(G)$, then both $N_{B_0'}(r_0)$ and $N_{B_p'}(r_{p+1})$ are cliques.*

Proof. Suppose that $N_{B_0'}(r_0)$ is not a clique. Let x, x' be two neighbors of r_0 in B_0' that are nonadjacent. By Observation 2, either $x = s_1$ or $x' = s_1$. Without loss of generality, we assume that $x' = s_1$.

If $r_0 = s_0$, then by our choice of B_0', $vs_1, vx \notin E(G)$ and the subgraph induced by $\{s_0, v, x, s_1\}$ is a claw, a contradiction. Thus we have that $r_0 = v$. If $s_1 s_{p+1} \notin E(G)$, then the subgraph induced by $\{v, s_{p+1}, x, s_1\}$ is a claw, a contradiction. Thus we assume that $s_1 s_{p+1} \in E(G)$. This implies $s_1 \in B_p$, $p = 1$, and so there are only two blocks of $G - v$. Note that $vs_1 \in E(G)$, so by our choice of B_1', $r_2 = v$. Thus we have $r_0 s_1 \in E(G)$ and $r_{p+1} s_p \in E(G)$. In particular, if $r_0 s_1 \notin E(G)$ or $r_{p+1} s_p \notin E(G)$, then $N_{B_0'}(r_0)$ is a clique, and by symmetry $N_{B_p'}(r_{p+1})$ is a clique too, proving the second statement of the observation.

Similarly, if we assume $N_{B_p'}(r_{p+1})$ is not a clique, we also get that $r_0 = r_{p+1} = v$, $p = 1$ and $vs_1 \in E(G)$.

Moreover, if neither $N_{B_0'}(r_0)$ nor $N_{B_p'}(r_{p+1})$ is a clique, then there is a neighbor x of v in $B_0 - s_1$ that is nonadjacent to s_1 and a neighbor y of v in $B_1 - s_1$ that is nonadjacent to s_1. But in that case the subgraph induced by $\{v, x, y, s_1\}$ is a claw, a contradiction. □

By Observation 3 and symmetry arguments, without loss of generality we may assume that $N_{B_p'}(r_{p+1})$ is a clique, and that the distance between r_0 and s_1 in B_0' is at least as large as between r_{p+1} and s_p in B_p'.

Let Q' be the (unique) path from r_0 to r_{p+1} (possibly consisting of one vertex v only) outside $B_0' \cup B_p'$. Note that Q and Q' are disjoint. We prove one more common observation.

Observation 4. *If the distance between r_{p+1} and s_p in B_p' is at least 4, then there is a neighbor of r_{p+1} outside B_p' that is nonadjacent to s_p.*

Proof. By our assumption, the distance between r_0 and s_1 in B'_0 is also at least 4. Let R' be a shortest path in B'_0 from r_0 to s_1. Then $R = Q'r_0R's_1Q$ is an induced path from r_{p+1} to s_p outside B'_p and of length at least 4. Let r'_{p+1} be the successor of r_{p+1} on R. Then $r'_{p+1}s_p \notin E(G)$. □

Now as in the set-up to Lemma 2, we set

$$N_i = \{v \in B'_0 - s_1 : d_{B'_0 - s_1}(v, r_0) = i\} \text{ and } j = \max\{i : N_i \neq \emptyset\}.$$

By Observation 2, N_1 is a clique. We complete the proof by assuming that there is no Hamilton path in B'_p from r_{p+1} to s_p, and showing that this implies that there exists a Hamilton path in B'_0 from r_0 to s_1. We start by proving the following claim on the structure of N_i.

Claim 1. $j \leq 2$ and N_2 is P_3-free.

Proof. If $j \geq 3$, then let x be a vertex in N_3, and let R' be a shortest path of $B'_0 - s_1$ from x to r_0. Then $R = Q'r_0R'$ is an induced path with origin r_{p+1} outside B'_p and of length at least 3. Using Lemma 3, we obtain a Hamilton path of B'_p from r_{p+1} to s_p. Hence, $j \leq 2$.

Let $xx'x''$ be an induced P_3 in N_2. Let w be a neighbor of x' in N_1. Then either wx or $wx'' \notin E(G)$; otherwise the subgraph induced by $\{w, r_0, x, x''\}$ is a claw. Without loss of generality, we assume that $wx'' \notin E(G)$. Then $R = Q'r_0wx'x''$ is an induced path with origin r_{p+1} outside B'_p and of length at least 3. Now Lemma 3 again implies that there is a Hamilton path of B'_p from r_{p+1} to s_p. Hence we conclude that N_2 is P_3-free. □

Claim 1 implies that every component of N_2 is a clique. To complete this subcase, we need one more observation on the existence of perfect paths.

Claim 2. Let H be a component of N_2. If s_1 is not adjacent to H, then H supports a perfect path to N_1; if s_1 is adjacent to H, then H supports a perfect path to N_1 and s_1.

Proof. We first assume that s_1 is not adjacent to H. If H contains only one vertex x, then by the 2-connectedness of G, x has at least two neighbors in N_1. Let w and w' be two neighbors of x in N_1. Then $R = wxw'$ is a perfect path of H to N_1.

If H contains at least two vertices, then by the 2-connectedness of G, H is joined to N_1 by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in H$ and $w, w' \in N_1$. Let R' be a Hamilton path of H from x to x'. Then $R = wxR'x'w'$ is a perfect path of H to N_1.

Suppose now that s_1 is adjacent to H. Let s' be a neighbor of s_1 in H. If H consists of the vertex s', then let w be a neighbor of s' in N_1. Then $R = ws's_1$ is a perfect path of H to N_1 and s_1. If there are at least two vertices in H, then let x be a vertex in H other than s'. Let w be a neighbor of x in N_1, and let R' be a Hamilton path of H from x to s'. Then $R = wxR's's_1$ is a perfect path of H to N_1 and s_1. □

Using Claim 2, by Lemma 2 we conclude that there exists a Hamilton path of B'_0 from r_0 to s_1, completing this case.

By the arguments in this section, it remains to complete the proofs of the three theorems only for the subcase that there is exactly one nontrivial inner block B_q and all the other blocks of $G - r$ are trivial. We do this separately for the three theorems in the following three sections.

7.5 Proof of Theorem 7.5

Let G be a 2-connected $\{K_{1,3}, B_{1,4}\}$-free graph. Adopting the notation and set-up of the previous section we are going to prove that G has a Hamilton path starting from a vertex r, in case $G - r$ contains a nontrivial inner block B_q and all other inner and end blocks of $G - r$ are trivial, so here we assume that all the blocks other than B_q are trivial.

Recall that it is sufficient to prove that B_q contains a Hamilton path from s_q to s_{q+1}. Suppose to the contrary that there is no such path. Set

$$N_i = \{v \in B_q - s_{q+1} : d_{B_q - s_{q+1}}(v, s_q) = i\}, \text{ and } j = \max\{i : N_i \neq \emptyset\}.$$

Note that $N_0 = \{s_q\}$ and $N_1 = N_{B_q}(s_q) \setminus \{s_{q+1}\}$.

We already know from Observation 1 that $N_{B_q}(s_q)$ is a clique and $N_{B_q}(s_{q+1})$ is a clique. In particular, this implies that N_1 is a clique. If $j = 1$, then let s' be a neighbor of s_{q+1} in N_1. If N_1 consists of the vertex s', then $R = s_q s' s_{q+1}$ is a Hamilton path of B_q from s_q to s_{q+1}, a contradiction. If N_1 contains at least two vertices, then let x be a vertex in N_1 other than s', and let R' be a

Hamilton path of N_1 from x to s'. Then $R = s_q x R' s' s_{q+1}$ is a Hamilton path of B_q from s_q to s_{q+1}, a contradiction. So there is nothing to prove if $N_2 = \emptyset$. Hence we assume $N_2 \neq \emptyset$. We complete the proof of this case by first proving a number of claims.

Claim 1. $rs_q \in E(G)$ and $rs_{q+1} \in E(G)$.

Proof. Suppose that $rs_q \notin E(G)$. Let Q be a shortest path from s_q to s_{p+1} containing rs_{p+1} with all internal vertices outside B_q. Then Q is an induced path of length at least 3 containing r with all internal vertices outside B_q.

Recall that N_1 is a clique. We first prove the following claim on the structure of N_i.

Claim 1.1. If N_2 is a clique, then for every i with $2 \leq i \leq j$, N_i is a clique.

Proof. We use induction on i. For $i = 2$, the assertion is true by assumption. Thus we assume that $3 \leq i \leq j$ and N_{i-1} is a clique.

Suppose that N_i is not a clique. Let x and x' be two vertices in N_i such that $xx' \notin E(G)$. If x and x' have a common neighbor in N_{i-1}, then let w be a common neighbor of x and x' in N_{i-1}, and y be a neighbor of w in N_{i-2}. Then the subgraph induced by $\{w, y, x, x'\}$ is a claw, a contradiction. Thus x and x' have no common neighbors in N_{i-1}.

Let w be a neighbor of x in N_{i-1} and w' be a neighbor of x' in N_{i-1}. Then from the above we conclude that $wx', w'x \notin E(G)$, and by the induction hypothesis, $ww' \in E(G)$. Let v be a neighbor of w in N_{i-2}. Then $vw' \in E(G)$; otherwise the subgraph induced by $\{w, v, w', x\}$ is a claw. Let R be a shortest path of $B_q - s_{q+1}$ from v to s_q. Then the subgraph induced by $\{w', w, x', x\} \cup V(R) \cup V(Q)$ is an $N_{1,1,\ell}$ with $\ell \geq 4$, so it contains an induced $B_{1,4}$, a contradiction. □

So, if N_2 is a clique, we can apply Lemma 2 and show the existence of a Hamilton path in B_q between s_q and s_{q+1}, a contradiction.

Hence, we assume next that N_2 is not a clique. We obtain more information on the structure of N_i by proving another set of claims.

Claim 1.2. If there is an induced P_3 in $\bigcup_{i=2}^{j} N_i$, then the level of the center vertex of the P_3 is larger than that of at least one of its end vertices.

Proof. Assuming the contrary, let $xx'x''$ be an induced P_3 in $\bigcup_{i=2}^{j} N_i$ such that x' is one of the vertices with the smallest level among the vertices in $\{x, x', x''\}$. Throughout the section, we call such a P_3 a *bad* P_3.

Suppose that $x' \in N_i$, where $i \geq 2$. Let w be a neighbor of x' in N_{i-1}. Then either wx or $wx'' \in E(G)$: otherwise the subgraph induced by $\{x', w, x, x''\}$ is a claw. Without loss of generality, we assume that $wx \in E(G)$. Then $wx'' \notin E(G)$; otherwise letting y be a neighbor of w in N_{i-2}, the subgraph induced by $\{w, y, x, x''\}$ is a claw.

Let R be a shortest path from w to s_q in $B_q - s_{q+1}$. Then the subgraph induced by $\{x, x', x''\} \cup V(R) \cup V(Q)$ is a $B_{1,\ell}$ with $\ell \geq 4$, a contradiction. □

Claim 1.3. N_2 is P_3-free and $\bigcup_{i=3}^{j} N_i$ is P_3-free.

Proof. If there is an induced P_3 in N_2, then it is a bad P_3, a contradiction to Claim 1.2. Thus N_2 is P_3-free.

Let $xx'x''$ be an induced P_3 in $\bigcup_{i=3}^{j} N_i$. Then by Claim 1.3, x' is not a vertex with the smallest level in $\{x, x', x''\}$. Without loss of generality, we assume that x has the smallest level. Moreover, we choose the induced P_3 in $\bigcup_{i=3}^{j} N_i$ subject to the other assumptions in such a way that the level of x is as small as possible.

We claim that $x \in N_3$. Assuming the contrary, suppose that $x \in N_i$, where $i \geq 4$. Then $x' \in N_{i+1}$. Let w be a neighbor of x in N_{i-1}. Clearly $wx' \notin E(G)$. Thus wxx' is an induced P_3 in $\bigcup_{i=3}^{j} N_i$ such that w has a smaller level than x, a contradiction to our choice of $xx'x''$. Thus as we claimed, $x \in N_3$ and then $x' \in N_4$.

Now let w be a neighbor of x in N_2. We have that $wx'' \notin E(G)$; otherwise letting y be a neighbor of w in N_1, the subgraph induced by $\{w, y, x, x''\}$ is a claw.

Let w' be a vertex in N_2 other than w. We claim that $ww' \in E(G)$. Assume the contrary. Note that w and w' have no common neighbors in N_1; otherwise letting v be a common neighbor of w and w' in N_1, the subgraph induced by $\{v, s_q, w, w'\}$ is a claw. Let now v be a neighbor of w in N_1 and v' be a neighbor of w' in N_1. Then $v'w \notin E(G)$ and the subgraph induced by $\{y', s_q, s_{q-1}, y, w, x, x', x''\}$ is a $B_{1,4}$, a contradiction. This implies that w is adjacent to all other vertices in N_2.

Let w', w'' be two vertices in N_2 other than w. We claim that $w'w'' \in E(G)$. Assume the contrary. If $w'x \in E(G)$, then by similar arguments as before we get that w' is adjacent to all other vertices in N_2, and then $w'w'' \in E(G)$. So we assume that $w'x \notin E(G)$ and similarly $w''x \notin E(G)$. Then the subgraph induced by $\{w, w', w'', x\}$ is a claw, a contradiction.

We conclude that N_2 is a clique, a contradiction. □

Claim 1.3 implies that every component of N_2 and $\bigcup_{i=3}^{j} N_i$ is a clique. Our next claims involve the connecting structure between such components.

Claim 1.4. Each component of N_2 is joined to at most one component of $\bigcup_{i=3}^{j} N_i$; each component of $\bigcup_{i=3}^{j} N_i$ is joined to at most two components of N_2.

Proof. Let C be a component of N_2 that is joined to at least two components D and D' of $\bigcup_{i=3}^{j} N_i$. Let R be a shortest path from D to D' with all internal vertices in C. Then R contains a bad P_3, a contradiction to Claim 1.2. Thus every component of N_2 is joined to at most one component of $\bigcup_{i=3}^{j} N_i$.

Let D be a component of $\bigcup_{i=3}^{j} N_i$ that is joined to at least three components C, C' and C'' of N_2. Let x, x' and x'' be three vertices of C, C' and C'', respectively, that are joined to D. Recall that any two vertices of $\{x, x', x''\}$ have no common neighbors in N_1. Let w, w' and w'' be the neighbors of x, x' and x'' in N_1, respectively.

If there is an induced path R of length at least 3 from x to x' with all internal vertices in D, then the subgraph induced by $\{w'', s_q, s_{q-1}, w\} \cup V(R)$ is an induced $B_{1,\ell}$ with $\ell \geq 4$, a contradiction. Thus we assume that all the induced paths from x to x' with all internal vertices in D have length 2. Hence x and x' have a common neighbor y in D. Similarly x' and x'' have a common neighbor y' in D.

If $x''y \in E(G)$, then the subgraph induced by $\{y, x, x', x''\}$ is a claw, a contradiction. So $x''y \notin E(G)$, and similarly $xy' \notin E(G)$, and the subgraph induced by $\{w, s_q, s_{q-1}, x, x', x'', y, y'\}$ is a $B_{1,4}$, a contradiction. □

Claim 1.5. Let H be a component of $\bigcup_{i=2}^{j} N_i$. If s_{q+1} is not joined to H, then H supports a perfect path to N_1; if s_{q+1} is joined to H, then H supports a perfect path to N_1 and s_{q+1}.

Proof. By Claim 1.4, one of the following situations applies to H:

(1) H consists of exactly one component C of N_2;

(2) H consists of one component C of N_2 and one component D of $\bigcup_{i=3}^{j} N_i$; or

(3) H consists of two components C and C' of N_2 and one component D of $\bigcup_{i=3}^{j} N_i$.

Case A. Situation (1) applies.

We first assume that s_{q+1} is not joined to H. If C has only one vertex x, then by the 2-connectedness of G, x has at least two neighbors in N_1. Let w, w' be two neighbors of x in N_1. Then $R = wxw'$ is a perfect path of H to N_1.

If C has at least two vertices, then by the 2-connectedness of G, C is joined to N_1 by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in C$ and $w, w' \in N_1$. Let R' be a Hamilton path of C from x to x'. Then $R = wxR'x'w'$ is a perfect path of H to N_1.

Suppose now that s_{q+1} is joined to H. Let s' be a neighbor of s_{q+1} in C. If C contains only the vertex s', then let w be a neighbor of s' in N_1. Then $R = ws's_{q+1}$ is a perfect path of H to N_1 and s_{q+1}.

If C contains at least two vertices, then let x be a vertex in C other than s', let w be a neighbor of x in N_1, and let R' be a Hamilton path of C from x to s'. Then $R = wxR's's_{q+1}$ is a perfect path of H to N_1 and s_{q+1}.

Case B. Situation (2) applies.

We first assume that s_{q+1} is not joined to H. Similarly as in the proof of Case A, D supports a perfect path R' to C. Let y and y' be the two end vertices of R'. By the 2-connectedness of G, C is joined to N_1 by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in C$ and $w, w' \in N_1$.

If x, x' and y, y' are distinct pairs, then without loss of generality, we assume that $x \neq y, y'$. If $x' \neq y, y'$, then let T be a path of C from x to y passing through all the vertices in $C \setminus \{x', y'\}$. Then $R = wxTyR'y'x'w'$ is a perfect path of H to N_1. If $x' = y$ or y', then without loss of generality, we assume that $x' = y'$. Let T be a path of C from x to y passing through all the vertices in $C \setminus \{x'\}$. Then $R = wxTyR'x'w'$ is a perfect path of H to N_1.

Now we assume that x, x' and y, y' are the same pair. If there is a third vertex x'' in C other that x and x', then let w'' be a neighbor of x'' in N_1. Without loss of generality, we assume that $w'' \neq w$. Then xw and $x''w''$ are two independent edges joining C to N_1 such that x, x'' and y, y' are distinct pairs. Then we can find a perfect path of H to N_1 in the same way as before. If we only have the vertices x and x' in C, then $R = wxR'x'w'$ is a perfect path of H to N_1.

Suppose now that s_{q+1} is joined to H. If s_{q+1} is joined to D, then let s' be a neighbor of s_{q+1} in D. If $|D| = 1$, the case is similar to Case A, hence we assume $|D| \geq 2$. By the 2-connectedness, not all vertices of C have the same common neighbor with s_{q+1} in D. This implies that we can choose s' in such a way that there is an edge zy with $z \in D \setminus \{s'\}$ and $y \in C$. Clearly, D supports a perfect path R' to C and s_{q+1} with end vertex y in C. If there is a second vertex x in C other than y, then let w be a neighbor of x in N_1 and let T be a Hamilton path of C from x to y. Then $R = wxTyR'$ is a perfect path of H to N_1 and s_{q+1}. If C has only one vertex y, then let w be a neighbor of y in N_1. Then $R = wyR'$ is a perfect path of H to N_1 and s_{q+1}.

Suppose now that s_{q+1} is not joined to D but joined to C. Let s' be a neighbor of s_{q+1} in C. Similarly as in the proof of Case A, D supports a perfect path R' to C. Let y and y' be the two end vertices of R'.

If there is a vertex x in C other than y, y' and s', then let w be a neighbor of x in N_1. If $s' \neq y, y'$, then let T be a path of C from x to y passing through all the vertices in $C \setminus \{y', s'\}$. Then $R = wxTyR'y's's_{q+1}$ is a perfect path of H to N_1 and s_{q+1}. If $s' = y$ or y', then without loss of generality, we assume that $s' = y'$. Let T be a path of C from x to y passing through all the vertices in $C \setminus \{y'\}$. Then $R = wxTyR'y's_{q+1}$ is a perfect path of H to N_1 and s_{q+1}.

Now we assume that there are no vertices in C other than y, y' and s'. If $s' \neq y, y'$, then let w be a neighbor of y in N_1. Then $R = wyR'y's's_{q+1}$ is a perfect path of H to N_1 and s_{q+1}. If $s' = y$ or y', then without loss of generality, we assume that $s' = y'$. Let w be a neighbor of y in N_1. Then $R = wyR'y's_{q+1}$ is a perfect path of H to N_1 and s_{q+1}.

Case C. Situation (3) applies.

We first assume that s_{q+1} is not joined to H. If D contains only one vertex y, then y has a neighbor in both C and C'. Let x and x' be the neighbors of y in C and C', respectively. Then $R' = xyx'$ is a perfect path of D to C and C'.

If D contains at least two vertices, then we claim that D is joined to C and C' by two independent edges. Let x and x' be two vertices in C and C', respectively, that are joined to D. If x and x' are joined to D by two independent edges, then clearly D is joined to C and C' by two independent edges. Thus we assume that x and x' are adjacent to only one common vertex y in D. Let y' be a neighbor of y in D. Then the subgraph induced by $\{y, x, x', y'\}$ is a claw, a contradiction. Thus, as we claimed, D is joined to C and C' by two independent edges. Let $yx, y'x'$ be two such edges, where

$y, y' \in D$, $x \in C$ and $x' \in C'$. Let R'' be a Hamilton path of D from y to y'. Then $R' = xyR''y'x'$ is a perfect path of D to C and C'. Thus in any case, D supports a perfect path R' to C and C'. Let x and x' be the two end vertices of R', where $x \in C$ and $x' \in C'$.

If C contains only the vertex x, then let $w = x$; otherwise let w be a vertex in C other than x. Let y be a neighbor of w in N_1, and let T be a Hamilton path of C from w to x. If C' contains only the vertex x', then let $w' = x'$; otherwise let w' be a vertex in C' other than x'. Let y' be a neighbor of w' in N_1, and let T' be a Hamilton path of C' from x' to w'. Note that C and C' have no common neighbors in N_1, so $y \neq y'$. Now $R = ywTxR'x'T'w'y'$ is a perfect path of H to N_1.

Suppose next that s_{q+1} is joined to H. If s_{q+1} is joined to C or C', then without loss of generality, we assume that s_{q+1} is joined to C', and that s' is a neighbor of s_{q+1} in C'. By similar arguments as before, there is a perfect path R' of D to C and C'. Let x and x' be the two end vertices of R', where $x \in C$ and $x' \in C'$. If C contains only the vertex x, then let $w = x$; otherwise let w be a vertex in C other than x. Let y be a neighbor of w in N_1, and let T be a Hamilton path of C from w to x. If $s' \neq x'$, then let T' be a Hamilton path of C' from x' to s'. Then $R = ywTxR'x'T's's_{q+1}$ is a perfect path of H to N_1 and s_{q+1}. Now we assume that $s' = x'$. If C' contains only the vertex x', then $R = ywTxR'x's'_{q+1}$ is a perfect path of H to N_1 and s_{q+1}. Thus we assume that C' contains a second vertex x'' other than x'. Let y' be a neighbor of x' in R'. Then we have that $x''y' \in E(G)$; otherwise $x''x'y'$ is a bad P_3, a contradiction to Claim 1.2. Thus $R' - y'x' \cup y'x''$ is a perfect path of D to C and C' such that $s' \neq x''$. Then we can find a perfect path of H to N_1 and s_{q+1} by similar arguments as before.

Suppose now that s_{q+1} is not joined to C and C', but that it is joined to D. Then s_{q+1} has no neighbors in any components of N_2 since $N_{B_q}(s_{q+1}) \setminus \{s_q\}$ is a clique and D cannot be joined to three components of N_2. Let x be a vertex in C joined to D, let w be a neighbor of x in N_1, let x' be a vertex in C' joined to D, and let w' be a neighbor of x' in N_1. Note that s_{q+1} has no neighbors in N_1 since $N_{B_q}(s_{q+1}) \setminus \{s_q\}$ is a clique, and $s_q s_{q+1} \notin E(G)$ since $N_{B_q}(s_q)$ is a clique. Thus the distance between s_q and s_{q+1} in B_q is at least 4. Note that s_{q-1} is a neighbor of s_q outside B_q and $s_{q-1}s_{q+1} \notin E(G)$. If the distance between x and s_{q+1} in $D \cup \{x, s_{q+1}\}$ is at least 3, then let R be a shortest path from x to s_{q+1} with all internal vertices in D. Then the subgraph induced by $\{w', s_q, s_{q-1}, w\} \cup V(R)$ is a $B_{1,\ell}$ with $\ell \geq 4$, a contradiction. Thus we assume that x and s_{q+1} have a common neighbor y in

D. Similarly, x' and s_{q+1} have a common neighbor y' in D. If $x'y \in E(G)$, then the subgraph induced by $\{y, x, x', s_{q+1}\}$ is a claw, a contradiction. Thus we assume that $x'y \notin E(G)$ and similarly $xy' \notin E(G)$. Then the subgraph induced by $\{s_{q+1}, y', x', y, x, w, s_q, s_{q-1}\}$ is a $B_{1,4}$, a contradiction. □

By Claim 1.5 we can apply Lemma 2 to obtain a Hamilton path of B_q from s_q to s_{q+1}, a contradiction.

Thus, $vs_q \in E(G)$. The second assertion follows by symmetry. □

We note here that in the above argumentation we have implicitly proved Lemma 3 in case $F = B_{1,4}$.

By Claim 1, $rs_q, rs_{q+1} \in E(G)$. If $p \geq 3$, G contains a claw centered at r, a contradiction. So $p = 2$, $q = 1$, and $G - r$ consists of three blocks. Recall that the two end blocks B_0 and B_2 are both trivial, so rs_0s_1r and rs_2s_3r are two triangles. We again obtain more information on the structure of N_i by proving the following claim.

Claim 2. $j \leq 3$, and N_3 is P_3-free.

Proof. If $j \geq 4$, then let x be a vertex in N_4, and let R be a shortest path from x to s_1 in $B_1 - s_2$. Then the subgraph induced by $\{s_0, r, s_3\} \cup V(R)$ is a $B_{1,4}$, a contradiction. Thus $j \leq 3$.

Let $xx'x''$ be an induced P_3 in N_3. Let w be a neighbor of x' in N_2, and let v be a neighbor of w in N_1. Then either wx or $wx'' \notin E(G)$; otherwise the subgraph induced by $\{w, v, x, x''\}$ is a claw. Without loss of generality, we assume that $wx'' \notin E(G)$. Then the subgraph induced by $\{s_0, r, s_3, s_1, v, w, x', x''\}$ is a $B_{1,4}$, a contradiction. □

The next claim shows that s_1 and s_2 are neighbors in B_1.

Claim 3. $s_1s_2 \in E(G)$.

Proof. Assuming the contrary, let d be the distance between s_1 and s_2 in B_1, and let Q be a shortest path from s_1 to s_2 in B_1. Then $d \geq 2$ and, since $j \leq 3$, we have $d \leq 4$. We distinguish three cases according to the value of d.

Case A. $d = 2$.

Let $Q = s_1xs_2$. If $G - x$ is 2-connected, then by the induction hypothesis, $G - x$ contains a Hamilton path P' starting from r. Clearly s_1 and s_2 are two cut vertices of $G-r$. Thus the subpath R' of P' from s_1 to s_2 is a Hamilton path of $B_1 - x$. Let s' be the neighbor of s_1 in R'. Then $xs' \in E(G)$ and $R = R' - s_1s' \cup s_1xs'$ is a Hamilton path of B_1 from s_1 to s_2, a contradiction. Thus there is another vertex y such that $\{x, y\}$ is a cut.

First note that $\{x, r\}$ is not a cut, since the only cut vertices of $G - v$ are s_1 and s_2. Thus $y \neq r$. Recalling that $s_1s_2 \notin E(G)$, by Lemma 1, s_1 and s_2 are not in a common component of $G-\{x,y\}$. Since s_1rs_2 is a path from s_1 to s_2 not passing through x, we have that either $y = s_1$ or $y = s_2$. Without loss of generality, we assume that $y = s_1$. Let H and H' be the two components of $G - \{x, s_1\}$, where $r \in H$. Let v be a vertex in H', and let R be an arbitrary path of G from v to s_3. Then R will pass through either x or s_1. Note that s_1 has only two neighbors r and s_0 in H. If R does not pass through x, then it will pass through either the edge s_1r or the subpath s_1s_0r. This implies that $\{x, r\}$ is a cut, a contradiction.

Case B. $d = 3$.

Let $Q = s_1xys_2$. Similarly as in Case A, we can prove that there is a vertex v such that $\{x, v\}$ is a cut, and $v \neq r$, s_1 or s_2. Since s_1 and y are both neighbors of x but $s_1y \notin E(G)$, they are not contained in the same component of $G - \{x, v\}$. Since s_1rs_2y is a path from s_1 to y not passing through x, we get that $v = y$.

Let H be the component of $G-\{x,y\}$ not containing r. Note that $N_{B_1}(s_1)$ and $N_{B_1}(s_2)$ are disjoint; otherwise we have $d = 2$. If x has a neighbor z outside $\{s_1\} \cup N_{B_1}(s_1) \cup H$, then let z' be a neighbor of x in H; in this case the subgraph induced by $\{x, s_1, z, z'\}$ is a claw, a contradiction. Thus all the neighbors of x are in $\{s_1\} \cup N_{B_1}(s_1) \cup H$, and similarly, all the neighbors of y are in $\{s_2\} \cup N_{B_1}(s_2) \cup H$. Let x' be a vertex in $N_{B_1}(s_1)$ other than x, and let y' be a vertex in $N_{B_1}(s_2)$ other than u. Then $y' \notin H$, hence $y'x \notin E(G)$.

If there is a vertex in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$, then without loss of generality, we assume that z is such a vertex and $zx' \in E(G)$. Then the subgraph induced by $\{s_3, v, s_0, s_2, u, x, x', z\}$ is a $B_{1,4}$, a contradiction. Thus we assume that there are no vertices in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$.

If H contains a vertex that is nonadjacent to x, then let z' be a vertex with distance 2 from x in H, and let z be a common neighbor of x and z' in H. Then the subgraph induced by $\{s_3, s_2, u', v, s_1, x, z, z'\}$ is a $B_{1,4}$, a contradiction. Thus we assume that every vertex in H is adjacent to x. Then by Lemma 1, H is a clique.

Let R' be a Hamilton path of $H \cup \{x, y\}$ from x to y, let T be a Hamilton path of $N_{B_1}(s_1)$ from x to x', and let T' be a Hamilton path of $N_{B_1}(s_2)$ from y to y'. Then $R = s_1x'TxR'yT'y's_2$ is a Hamilton path of B_1 from s_1 to s_2, a contradiction.

Case C. $d = 4$.

Let $Q = s_1xyzs_2$. Similarly as in Case B, we have that either $\{x, y\}$ or $\{x, z\}$ is a cut. We claim that $\{x, z\}$ is a cut. Assuming the contrary, $\{x, y\}$ is a cut, and similarly $\{y, z\}$ is a cut. Let H be the component of $G - \{x, y\}$ not containing r, and let H' be the component of $G - \{y, z\}$ not containing r. If H and H' share a common vertex v, then there is a path between x and z through v with all internal vertices in $H \cup H'$, implying that r is in the same component of $G - \{y, z\}$ as v, a contradiction. So H and H' are disjoint. Then every neighbor of y is in either $H \cup \{x\}$ or $H' \cup \{z\}$. Thus every path of G from y to r passes through either x or z, and then $\{x, z\}$ is a cut, a contradiction.

Let x' be a vertex in $N_{B_1}(s_1)$ other than x. Then $x'y \notin E(G)$ and the subgraph induced by $\{s_3, v, s_0, s_2, z, y, x, x'\}$ is a $B_{1,4}$, a contradiction. □

By Observation 1 and Claim 3, $N_{B_1}(s_2) \setminus \{s_1\} = N_{B_1}(s_1) \setminus \{s_2\} = N_1$.

Our next claim shows that the vertices of N_1 can be paired into vertex cuts, as follows.

Claim 4. For every vertex $x \in N_1$, there is a unique vertex $x' \in N_1 \setminus \{x\}$ such that $\{x, x'\}$ is a cut.

Proof. Assume that there are no such vertices. Similarly as in the proof of Claim 3, we have that x is contained in a cut $\{x, y\}$ with $y \neq r$, s_1 or s_2. It is easy to check that $y \neq s_0$ or s_3. Thus $y \in \bigcup_{i=2}^{j} N_i$. Let H be the component of $G - \{x, y\}$ not containing r, and let Q be a shortest path from x to y with all internal vertices in H.

Let R be a shortest path in $G - x$ from y to N_1, and let x' be the end vertex of R other than y. Similarly as in the proof of Claim 3, x' is contained in a cut $\{x', y'\}$ with a vertex $y' \neq s_1$. Let z' be the neighbor of x' in R. Note that s_1 and z' are not contained in a common component of $G - \{x', y'\}$. Note that $s_1xQ \cup R - z'x'$ is a path from s_1 to z' not passing through x'. We have that y' must be a vertex in $V(Q) \cup V(R) \setminus \{x'\}$. By our assumption $y' \neq x$. If $y' \in H \cup \{y\}$, then let H' be the component of $G - \{x', y'\}$ not containing r. Then every neighbor of y will be either in $H \cup \{x\}$ or in $H' \cup \{x'\}$. Hence every path from y to r will pass through either x or x', a contradiction. Thus $y' \in V(R) \setminus \{x', y\}$.

Let T be the subpath of R from y to y', let H' be the component of $G - \{x', y'\}$ not containing r, and let z' be a neighbor of y' in H'. Then the subgraph induced by $\{s_0, r, s_3\} \cup V(Q) \cup V(T) \cup \{z'\}$ is a $B_{1,\ell}$ with $\ell \geq 4$, a contradiction.

Thus we conclude that there is a vertex $x' \in N_1$ such that $\{x, x'\}$ is a cut.

Let H be the component of $G - \{x, x'\}$ not containing r. We have that all the neighbors of x in $\bigcup_{i=2}^{j} N_i$ are in H; otherwise, let y be a neighbor of x in H, and let y' be a neighbor of x in $\bigcup_{i=2}^{j} N_i \setminus H$. Then the subgraph induced by $\{x, s_1, y, y'\}$ is a claw. This implies that for any vertex x'' in $N_1 \setminus \{x, x'\}$, the pair $\{x, x''\}$ is not a cut. □

By Claim 4, we can partition N_1 into pairs such that each pair is a cut. The next claim shows how we can pick up the vertices of components in paths between the pairs.

Claim 5. Let $\{t, t'\}$ be a cut of G such that $t, t' \in N_1$, and let H be the component of $G - \{t, t'\}$ not containing r. Then there is a perfect path of H to $\{t, t'\}$.

Proof. If $H \cap N_2$ contains only one vertex x, then by the 2-connectedness of G, $H \cap N_3 = \emptyset$ and $xt, xt' \in E(G)$. Then $R = txt'$ is a perfect path of H to $\{t, t'\}$. Next we assume that $H \cap N_2$ contains at least two vertices. Note that both t and t' are adjacent to some vertices in $H \cap N_2$. We can divide $H \cap N_2$ into two nonempty subsets C and C' such that every vertex in C is adjacent to t, and every vertex in C' is adjacent to t'.

Recall that $j \leq 3$ and N_3 is P_3-free, so every component of $H \cap N_3$ is a clique.

Claim 5.1. Let D be a component of $H \cap N_3$. If D is joined to C but not to C', then D supports a perfect path to C; if D is joined to C' but not to C, then D supports a perfect path to C'; and if D is joined to both C and C', then D contains a perfect path to C and C'.

Proof. **Case A.** D is joined to C but not to C'.

If D contains only one vertex x, then by the 2-connectedness of G, x has at least two neighbors in C. Let w, w' be two neighbors of x in C. Then $R = wxw'$ is a perfect path of D to C.

Now we assume that D contains at least two vertices. By the 2-connectedness of G, D is joined to C by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in D$ and $w, w' \in C$. Let R' be a Hamilton path of D from x to x'. Then $R = wxR'x'w'$ is a perfect path of D to C.

Case B. D is joined to C' but not to C.

This case can be treated in a similar way as Case A.

Case C. D is joined to both C and C'.

If D consists of the vertex x, then x has at least one neighbor in C and in C'. Let w be a neighbor of x in C, and let w' be a neighbor of x in C'. Then $R = wxw'$ is a perfect path of D to C and C'.

Now we assume that D contains at least two vertices. Clearly D is joined to C and C' by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in D$, $w \in C$ and $w' \in C'$. Let R' be a Hamilton path of D from x to x'. Then $R = wxR'x'w'$ is a perfect path of D to C and C'. $\square$

Let $\mathcal{D} = \{D_1, D_2, \ldots, D_k\}$ be the set of components in $H \cap N_3$ that are joined to C but not to C', let R_i $(1 \le i \le k)$ be a perfect path of D_i to C, and let x_i, y_i be the two end vertices of R_i; let $\mathcal{D}' = \{D'_1, D'_2, \ldots, D'_{k'}\}$ be the set of components in $H \cap N_3$ that are joined to C' but not to C, let R'_i $(1 \le i \le k')$ be a perfect path of D'_i to C', and let x'_i, y'_i be the two end vertices of R'_i; let $\mathcal{D}'' = \{D''_1, D''_2, \ldots, D''_{k''}\}$ be the set of components in $H \cap N_3$ that are joined to both C and C', let R''_i $(1 \le i \le k'')$ be a perfect path of D''_i to C and C', and let x''_i, y''_i be the two end vertices of R''_i, where $x''_i \in C$ and $y''_i \in C'$.

We first assume that k'' is odd. If $\mathcal{D} \neq \emptyset$, then let $w = x_1$; otherwise let $w = x''_1$. Let T be a path from t to w passing through all the vertices in $C \setminus \bigcup_{i=1}^{k} \{x_i, y_i\} \setminus \bigcup_{i=1}^{k''} \{x''_i\}$. If $\mathcal{D}' \neq \emptyset$, then let $w' = y'_{k'}$; otherwise let $w' = y''_{k''}$. Let T' be a path from t' to w' passing through all the vertices in $C' \setminus \bigcup_{i=1}^{k'} \{x'_i, y'_i\} \setminus \bigcup_{i=1}^{k''} \{y''_i\}$. Then $R = Tx_1R_1y_1 \cdots x_kR_ky_kx''_1R''_1y''_1y''_2R''_2x''_2 \cdots x''_{k''}R''_{k''}y''_{k''}x'_1R'_1y'_1 \cdots x'_{k'}R'_{k'}y'_{k'}T'$ is a perfect path of H to $\{t, t'\}$.

Next we assume that k'' is even. If there is an edge joining C to C' such that its two vertices are not the two end vertices of a common perfect path of some component in $\mathcal{D}''$ (we call such an edge a *good edge*), then let zz' be a good edge, where $z \in C$ and $z' \in C'$. Note that z is possibly an end vertex of a perfect path of some component in $\mathcal{D}$ or $\mathcal{D}''$, or that it is not such an end vertex, and that z' is possibly an end vertex of a perfect path of some component in $\mathcal{D}'$ or $\mathcal{D}''$, or that it is not such an end vertex. So there are nine different cases to consider. Here we only discuss two of the cases; for the other cases, a perfect path of H to $\{t,t'\}$ can be found in a similar way.

If z is not an end vertex of a perfect path of some component in $\mathcal{D}$ or $\mathcal{D}''$, and z' is an end vertex of a perfect path of some component in $\mathcal{D}'$, then without loss of generality, we assume that $z' = x'_1$. If $\mathcal{D} \neq \emptyset$, then let $w = x_1$; otherwise, if $\mathcal{D}'' \neq \emptyset$, then let $w = x''_1$; otherwise let $w = z$. Let T be a path from t to w passing through all the vertices in $C \setminus \bigcup_{i=1}^{k}\{x_i, y_i\} \setminus \bigcup_{i=1}^{k''}\{x''_i\} \setminus \{z\}$. Let T' be a path from t' to $y'_{k'}$ passing through all the vertices in $C' \setminus \bigcup_{i=1}^{k'}\{x'_i, y'_i\} \setminus \bigcup_{i=1}^{k''}\{y''_i\}$. Then $R = Tx_1R_1y_1 \cdots x_kR_ky_kx''_1R''_1y''_1y''_2R''_2x''_2 \cdots y''_{k''}R''_{k''}x''_{k''}\ zx'_1R'_1y'_1 \cdots x'_{k'}R'_{k'}y'_{k'}T'$ is a perfect path of H to $\{t,t'\}$.

If both z and z' are end vertices of perfect paths of some components in $\mathcal{D}''$, then note that zz' is a good edge, so these vertices are not the end vertices of a common perfect path. Without loss of generality, we assume that $z = x''_2$ and $z' = y''_1$. If $\mathcal{D} \neq \emptyset$, then let $w = x_1$; otherwise let $w = x''_1$. Let T be a path from t to w passing through all the vertices in $C \setminus \bigcup_{i=1}^{k}\{x_i, y_i\} \setminus \bigcup_{i=1}^{k''}\{x''_i\}$. If $\mathcal{D}' \neq \emptyset$, then let $w' = y'_{k'}$; otherwise let $w' = y''_{k''}$. Let T' be a path from t' to w' passing through all the vertices in $C' \setminus \bigcup_{i=1}^{k'}\{x'_i, y'_i\} \setminus \bigcup_{i=1}^{k''}\{y''_i\}$. Then $R = Tx_1R_1y_1 \cdots x_kR_ky_kx''_1R''_1y''_1x''_2R''_2y''_2 \cdots x''_{k''}R''_{k''}y''_{k''}x'_1R'_1y'_1 \cdots x'_{k'}R'_{k'}y'_{k'}T'$ is a perfect path of H to $\{t,t'\}$.

Next we assume that each edge joining C to C' is not a good edge.

If C is not joined to C', then $\mathcal{D}'' \neq \emptyset$; otherwise t will be a cut vertex of G. If C is joined to C', then we also have $\mathcal{D}'' \neq \emptyset$, since every edge joining C to C' is not good. Recall that we assume that k'' is even, so we have $k'' \geq 2$.

Note that $x''_1y''_2, x''_2y''_1 \notin E(G)$; otherwise they are good edges. Thus $ty''_1, ty''_2 \notin E(G)$; otherwise the subgraph induced by $\{t, s_1, x''_2, y''_1\}$ or $\{t, s_1, x''_1, y''_2\}$ is a claw. Let R be a shortest path from x''_1 to y''_1 with all internal vertices in D''_1 (possibly of length 1). Then the subgraph induced by $\{s_0, r, s_3, s_1, t\} \cup V(R) \cup \{y''_2\}$ is a $B_{1,\ell}$ with $\ell \geq 4$, a contradiction. □

Let $N_1 = \{x_i, x'_i : 1 \leq i \leq k\}$ such that for every i with $1 \leq i \leq k$, $\{x_i, x'_i\}$ is a cut. Let H_i be the component of $G - \{x_i, x'_i\}$ not containing v, and let R_i be a perfect path of H_i to $\{x_i, x'_i\}$. Then $R = s_1x_1R_1x'_1 \cdots x_kR_kx'_ks_2$ is a Hamilton path of B_1 from s_1 to s_2, our final contradiction.

7.6 Proof of Theorem 7.6

Let G be a 2-connected $\{K_{1,3}, B_{2,3}\}$-free graph. Adopting the notation and set-up of Section 7.4 we are going to prove that G has a Hamilton path starting from a vertex v, in case $G-r$ contains a nontrivial inner block B_q and all other inner and end blocks of $G-r$ are trivial. Recall that it is sufficient to prove that B_q contains a Hamilton path from s_q to s_{q+1}. Suppose to the contrary that there is no such path. Set

$$N_i = \{v \in B_q - s_{q+1} : d_{B_q - s_{q+1}}(v, s_q) = i\}, \text{ and } j = \max\{i : N_i \neq \emptyset\}.$$

Note that $N_0 = \{s_q\}$ and $N_1 = N_{B_q}(s_q) \setminus \{s_{q+1}\}$.

We already know from Observation 1 that $N_{B_q}(s_q)$ is a clique and $N_{B_q}(s_{q+1})$ is a clique. In particular, this implies that N_1 is a clique. If $N_2 = \emptyset$, there is nothing to prove, so we assume $N_2 \neq \emptyset$. We complete the proof of this case by first proving a number of claims.

Claim 1. $rs_q \in E(G)$ and $rs_{q+1} \in E(G)$.

Proof. Suppose that $rs_q \notin E(G)$. Let Q be a shortest path from s_q to s_{p+1} containing rs_{p+1} and all internal vertices outside B_q. Then Q is an induced path containing r with all internal vertices outside B_q and of length at least 3.

We consider the structure of N_i and prove the following claim.

Claim 1.1. For every i with $1 \leq i \leq j-1$, N_i is a clique, and N_j is P_4-free.

Proof. We use induction on i. We already know that N_1 is a clique, so we assume that $2 \leq i \leq j-1$.

Let x be a vertex in N_i that has a neighbor y in N_{i+1}. Let x' be a vertex in N_i other than x. We first claim that $xx' \in E(G)$. Assume the contrary. Then x and x' have no common neighbors in N_{i-1}. Let w be a neighbor of x in N_{i-1}, and let w' be a neighbor of x' in N_{i-1}. Then $wx', w'x \notin E(G)$, and by the induction hypothesis, $ww' \in E(G)$. Let v be a neighbor of w in N_{i-2}. Then $vw' \in E(G)$; otherwise the subgraph induced by $\{w, v, w', x\}$ is a claw. Let R be a shortest path of $B_q - s_{q+1}$ from v to s_q. Then the subgraph induced by $\{w', w, x, y\} \cup V(R) \cup V(Q)$ is a $B_{2,\ell}$ with $\ell \geq 3$, a contradiction. Thus, as we claimed, x is adjacent to all other vertices in N_i.

Let x', x'' be two arbitrary vertices in N_i other than x. We claim that $x'x'' \in E(G)$. Assume the contrary. If $x'y \in E(G)$, then similarly as before, x' is adjacent to all other vertices in N_i and $x'x'' \in E(G)$. Thus we assume that $x'y \notin E(G)$ and similarly $x''y \notin E(G)$. Then the subgraph induced by $\{x, x', x'', y\}$ is a claw, a contradiction.

Thus N_i is a clique.

Let $xx'x''x'''$ be an induced P_4 in N_j. Let w be a neighbor of x in N_{j-1}, and let w''' be a neighbor of x''' in N_{j-1}. Then $wx'' \notin E(G)$; otherwise let v be a neighbor of w in N_{j-2}. Then the subgraph induced by $\{w, v, x, x''\}$ is a claw. Similarly, $wx''', w'''x, w'''x' \notin E(G)$. If $wx' \in E(G)$, then let R be a shortest path of $B_q - s_{q+1}$ from w to s_q. Then the subgraph induced by $\{x, x', x'', x'''\} \cup V(R) \cup V(Q)$ is a $B_{2,\ell}$ with $\ell \geq 3$, a contradiction. Thus we assume that $wx' \notin E(G)$, and similarly $w'''x'' \notin E(G)$. Let v be a neighbor of w in N_{j-2}. Then $w'''v \in E(G)$; otherwise the subgraph induced by $\{w, v, w''', x\}$ is a claw. Let R be a shortest path of $B_q - s_{q+1}$ from u to s_q. Then the subgraph induced by $\{w''', w, x, x'\} \cup V(R) \cup V(Q)$ is a $B_{2,\ell}$ with $\ell \geq 3$, a contradiction.

Thus N_j is P_4-free. □

We next prove the following claim on the existence of perfect paths.

Claim 1.2. Let H be a component of N_j. If s_{q+1} is not adjacent to a vertex of H, then H supports a perfect path to N_{j-1}; if s_{q+1} is adjacent to a vertex of H, then H contains a perfect path to N_{j-1} and s_{q+1}.

Proof. We distinguish three cases.

Case A. H contains only one or two vertices.

We first assume that s_{q+1} is not adjacent to H. If H contains only one vertex x, then by the 2-connectedness of G, x has at least two neighbors in N_{j-1}. Let w and w' be two neighbors of x in N_{j-1}. Then $R = wxw'$ is a perfect path of H to N_{j-1}. If H contains two vertices x and x', then by the 2-connectedness of G, x and x' are joined to N_{j-1} by two independent edges. Let xw and $x'w'$ be two such edges. Then $R = wxx'w'$ is a perfect path of H to N_{j-1}.

Suppose now that s_{q+1} is adjacent to H. If H contains only one vertex x, then x is adjacent to s_{q+1}. Let w be a neighbor of x in N_{j-1}. Then $R = wxs_{q+1}$ is a perfect path of H to N_{j-1} and s_{q+1}. If H contains two vertices x and x', then without loss of generality, we assume that $x's_{q+1} \in E(G)$. Let w be a neighbor of x in N_{j-1}. Then $R = wxx's_{q+1}$ is a perfect path of H to N_{j-1} and s_{q+1}.

Case B. H is 2-connected.

We use that N_j is P_4-free, and thus H is P_4-free and also N-free. By Theorem 7.1, H contains a Hamilton cycle C.

We first assume that s_{q+1} is not adjacent to H. By the 2-connectedness of G, not all the vertices of H are adjacent to only one common vertex in N_{j-1}. Thus there are two vertices x and x' of H that are adjacent on C such that x and x' are joined to N_{j-1} by two independent edges. Let w and w' be the neighbors of x and x' in N_{j-1} such that $w \neq w'$. Then $R = C - xx' \cup \{xw, x'w'\}$ is a perfect path of H to N_{j-1}.

Suppose now that s_{q+1} is adjacent to H. Let s' be a neighbor of s_{q+1} in H, let x be a vertex in H that is adjacent to s' on C, and let w be a neighbor of x in N_{j-1}. Then $R = C - xs' \cup \{xw, s's_{q+1}\}$ is a perfect path of H to N_{j-1} and s_{q+1}.

Case C. H has a cut vertex.

Let x be a cut vertex of H. Obviously, $H - x$ has exactly two components. Let C and C' be the two components of $H - x$. If there is a vertex in C that is nonadjacent to x, then let z be a vertex in C with distance 2 from x in C, let y be a common neighbor of x and z in C, and let y' be a neighbor of x in C'. Then $zyxy'$ is an induced P_4 in H, a contradiction. This implies that x is adjacent to every vertex in C. If there are two vertices y, z in C that are nonadjacent, then let y' be a neighbor of x in C'; then the subgraph induced by $\{x, y, z, y'\}$ is a claw, a contradiction. Thus $C \cup \{x\}$ is a clique and similarly $C' \cup \{x\}$ is a clique.

We first assume that s_{q+1} is not adjacent to H. Let y be a vertex in C and let y' be a vertex in C'. Let T be a Hamilton path of $C \cup \{x\}$ from x to y, let w be a neighbor of y in N_{j-1}, let T' be a Hamilton path of $C' \cup \{x\}$ from x to y', and let w' be a neighbor of y' in N_{j-1}. Then $R = wyTxT'y'w'$ is a perfect path of H to N_{j-1}.

Suppose now that s_{q+1} is adjacent to H. We claim that s_{q+1} must be adjacent to C or C'. Assuming the contrary, s_{q+1} has only one neighbor x in H. Let y be a vertex in C, and let y' be a vertex in C'. Then the subgraph induced by $\{x, y, y', s_{q+1}\}$ is a claw, a contradiction. Without loss of generality, we assume that s_{q+1} is adjacent to C'. Let s' be a neighbor of s_{q+1} in C', and let y be a vertex in C. Let T be a Hamilton path of $C \cup \{x\}$ from x to y, let

w be a neighbor of y in N_{j-1}, and let T' be a Hamilton path of $C' \cup \{x\}$ from x to s'. Then $R = wyTxT's's_{q+1}$ is a perfect path of H to N_{j-1} and s_{q+1}. □

The above claims and Lemma 2 imply that there exists a Hamilton path of B_q from s_q to s_{q+1}, a contradiction. Thus we conclude that $rs_q \in E(G)$. The second assertion follows by symmetry. □

We note here that in the above argumentation we have implicitly proved Lemma 3 in case $F = B_{2,3}$.

By Claim 1, $rs_q, rs_{q+1} \in E(G)$. If $p \geq 3$, G contains a claw centered at r, a contradiction. So $p = 2$, $q = 1$, and $G - r$ consists of three blocks. Recall that the two end blocks B_0 and B_2 are both trivial, so rs_0s_1r and rs_2s_3r are two triangles. We again obtain more information on the structure of N_i by proving the following claims.

Claim 2. $j \leq 3$, and N_3 is P_3-free.

Proof. The proofs of the following implications are completely analogous to the proofs of Claim 1.1 and 1.2, and the application of Lemma 2, and are therefore omitted.

Claim 2.1. If N_2 is a clique, then for every i with $2 \leq i \leq j-1$, N_i is a clique and N_j is P_4-free.

Claim 2.2. If for every i with $1 \leq i \leq j-1$, N_i is a clique and N_j is P_4-free, then B_1 contains a Hamilton path from s_1 to s_2.

Thus if N_2 is a clique, then by Claims 2.1 and 2.2, there is a Hamilton path of B_1 from s_1 to s_2, a contradiction. So we assume that N_2 is not a clique.

If $j \geq 4$, then let z be a vertex in N_4, let y be a neighbor of z in N_3, and let x be a neighbor of y in N_2. Let x' be a vertex in N_2 other than x. We claim that $xx' \in E(G)$. Assume the contrary. Then x and x' have no common neighbors in N_1. Let w be a neighbor of x in N_1, and let w' be a neighbor of x' in N_1. Then $w'x \notin E(G)$, and the subgraph induced by $\{w', s_1, r, s_3, w, x, y, z\}$ is a $B_{2,3}$, a contradiction. This implies that x is adjacent to all the other vertices in N_2.

Now let x' and x'' be two vertices in N_2 other than x. We claim that $x'x'' \in E(G)$. Assume the contrary. If $x'y \in E(G)$, then similarly as before, x' is adjacent to all the other vertices in N_2, and then $x'x'' \in E(G)$. Thus we assume that $x'y \notin E(G)$, and similarly $x''y \notin E(G)$. Then the subgraph induced by $\{x, x', x'', y\}$ is a claw, a contradiction.

This implies that N_2 is a clique, a contradiction. Thus $j \leq 3$.

Let $yy'y''$ be an induced P_3 in N_3. Let x be a neighbor of y' in N_2. Then x is nonadjacent to y or y''; otherwise, let w be a neighbor of x in N_1; then the subgraph induced by $\{x, w, y, y''\}$ is a claw. Without loss of generality, we assume that $xy'' \notin E(G)$. Then similarly as before, we can prove that N_2 is a clique, a contradiction. Thus N_3 is P_3-free. □

We next show that s_1 and s_2 are neighbors in B_1.

Claim 3. $s_1s_2 \in E(G)$.

Proof. Assume the contrary. Let d be the distance between s_1 and s_2 in B_1 and let Q be a shortest path from s_1 to s_2 in B_1. Then $d \geq 2$ and, since $j \leq 3$, we have $d \leq 4$. We distinguish three cases according to the value of d.

Case A. $d = 2$.

Noting that we have not used $B_{1,4}$-freeness in Case A of the proof of Claim 3 in Section 7.5, this case can be proved completely analogously.

Case B. $d = 3$.

Let $Q = s_1xys_2$. Similarly as in Case B of the proof of Claim 3 in Section 7.5, we can prove that $\{x, y\}$ is a cut of G. Note that $N_{B_1}(s_1)$ and $N_{B_1}(s_2)$ are disjoint; otherwise $d = 2$. Let H be the component of $G - \{x, y\}$ not containing r. Let x' be a vertex in $N_{B_1}(s_1)$ other than x, and let y' be a vertex in $N_{B_1}(s_2)$ other than y.

If there is a vertex in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$, then without loss of generality, we assume that z is such a vertex and $zx' \in E(G)$. Let z' be a neighbor of y in H. Then the subgraph induced by $\{s_0, s_1, x', z, r, s_2, y, z'\}$ is a $B_{2,3}$, a contradiction. Thus we assume that there are no vertices in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$.

If H contains a vertex nonadjacent with x, then let z' be a vertex with distance 2 from x in H, and let z be a common neighbor of x and z' in H. Then the subgraph induced by $\{s_0, r, s_2, y', s_1, x, z, z'\}$ is a $B_{2,3}$, a contradiction. Thus we assume that every vertex in H is adjacent to x. Then by Lemma 1, H is a clique.

Let R' be a Hamilton path of $H \cup \{x, y\}$ from x to y, let T be a Hamilton path of $N_{B_1}(s_1)$ from x to x', and let T' be a Hamilton path of $N_{B_1}(s_2)$ from

y to y'. Then $R = s_1x'TxR'yT'y's_2$ is a Hamilton path of B_1 from s_1 to s_2, a contradiction.

Case C. $d = 4$.

Let $Q = s_1xyzs_2$. Similarly as in Case C of the proof of Claim 3 in Section 7.5, we can prove that $\{x, z\}$ is a cut of G. Note that $N_{B_1}(s_1)$ and $N_{B_1}(s_2)$ are disjoint and not adjacent; otherwise $d \leq 3$. Let x' be a vertex in $N_{B_1}(s_1)$ other than x, and let z' be a vertex in $N_{B_1}(s_2)$ other than z. Then the subgraph induced by $\{x', s_1, r, s_3, x, y, z, z'\}$ is a $B_{2,3}$, a contradiction. □

By Observation 1 and Claim 3, $N_{B_1}(s_2)\setminus\{s_1\} = N_{B_1}(s_1)\setminus\{s_2\} = N_1$. Our next observation shows that N_1 can be partitioned into cut pairs.

Claim 4. For every vertex $x \in N_1$, there is unique vertex $x' \in N_1\setminus\{x\}$ such that $\{x, x'\}$ is a cut.

Proof. Assume the contrary. Similarly as in the proof of Claim 4 in Section 7.5, there is a vertex $y \in \bigcup_{i=2}^{j} N_i$ such that $\{x, y\}$ is a cut. Let H be the component of $G - \{x, y\}$ not containing r, and let R be a shortest path from x to y with all internal vertices in H.

Let R' be a shortest path in $G - x$ from y to N_1, and let x' be the end vertex of R' other than y. Similarly as in the proof of Claim 4 in Section 7.5, x' is contained in a cut $\{x', y'\}$, and with the other vertex $y' \in V(R') \setminus \{x', y\}$. Let T' be the subpath of R' from y to y', and let H' be the component of $G - \{x', y'\}$ not containing r.

Note that $\{x, x'\}$ is not a cut by our assumption. Let R'' be a shortest path of $G - \{x, x'\}$ from T' to N_1, and let x'' be the end vertex of R'' in N_1. Similarly as before, we have that x'' is contained in a cut $\{x'', y''\}$, and with the other vertex $y'' \in V(R'') \setminus \{x'', y, y'\}$. Let H'' be the component of $G - \{x'', y''\}$ not containing r.

If T' passes through y'', then let z and z' be the two neighbors of y'' on T', and let z'' be a neighbor of y'' in H''. Then the subgraph induced by $\{y'', z, z', z''\}$ is a claw, a contradiction. Thus we assume that T' does not pass through y''.

Let z' be a neighbor of y' in H'. Then the subgraph induced by $\{x'', s_1, r, s_3\} \cup V(R) \cup V(T') \cup \{z'\}$ is a $B_{2,\ell}$ with $\ell \geq 3$, a contradiction.

Thus there is a vertex $x' \in N_1$ such that $\{x, x'\}$ is a cut.

One can prove the uniqueness similarly as in the proof of Claim 4 in Section 7.5. □

By Claim 4, we can partition N_1 into pairs such that each pair is a cut. These pairs have a nice property with respect to perfect paths, as follows.

Claim 5. Let $\{t, t'\}$ be a cut of G such that $t, t' \in N_1$, and let H be the component of $G - \{t, t'\}$ not containing r. Then there is a perfect path of H to $\{t, t'\}$.

Proof. If $H \cap N_2$ contains only one vertex x, then by the 2-connectedness of G, $H \cap N_3 = \emptyset$ and $xt, xt' \in E(G)$. Then $R = txt'$ is a perfect path of H to $\{t, t'\}$. Thus we assume that $H \cap N_2$ contains at least two vertices. Note that both t and t' are adjacent to some vertices in $H \cap N_2$. We can divide $H \cap N_2$ into two nonempty subsets C and C' such that every vertex of C is adjacent to t and every vertex of C' is adjacent to t'.

Recall that $j \leq 3$ and that N_3 is P_3-free, so every component of $H \cap N_3$ is a clique. The proof of the next observations is completely analogous to the proof of Claim 5.1 in Section 7.5.

Claim 5.1. Let D be a component of $H \cap N_3$. If D is joined to C but not to C', then D supports a perfect path to C; if D is joined to C' but not to C, then D supports a perfect path to C'; and if D is joined to both C and C', then D contains a perfect path to C and C'.

We proceed similarly as in Section 7.5.

Let $\mathcal{D} = \{D_1, D_2, \ldots, D_k\}$ be the set of components in $H \cap N_3$ that are joined to C but not to C', let R_i $(1 \leq i \leq k)$ be a perfect path of D_i to C, and let x_i, y_i be the two end vertices of R_i; let $\mathcal{D}' = \{D'_1, D'_2, \ldots, D'_{k'}\}$ be the set of components in $H \cap N_3$ that are joined to C' but not to C, let R'_i $(1 \leq i \leq k')$ be a perfect path of D'_i to C', and let x'_i, y'_i be the two end vertices of R'_i; and let $\mathcal{D}'' = \{D''_1, D''_2, \ldots, D''_{k''}\}$ be the components in $H \cap N_3$ that are joined to both C and C', let R''_i $(1 \leq i \leq k'')$ be a perfect path of D''_i to C and C', and let x''_i, y''_i be the two end vertices of R''_i, where $x''_i \in C$ and $y''_i \in C'$.

If k'' is odd, or k'' is even and there is a good edge joining C to C', then we can prove the assertion similarly as in Section 7.5. Thus we assume that k'' is even and that every edge joining C to C' is not good. Similarly as in Section 7.5, note that $k'' \geq 2$.

If C is joined to C', then without loss of generality, we assume that $x_1''y_1'' \in E(G)$. Let z be a neighbor of x_1'' in D_1'', and let z' be a neighbor of y_2'' in D_2''. Then we have $x_1''y_2'', x_2'', y_1'', ty_1'', ty_2'' \notin E(G)$. Besides, we have that $y_1''z \in E(G)$; otherwise the subgraph induced by $\{x_1'', t, y_1'', z\}$ is a claw. Thus the subgraph induced by $\{z, y_1'', y_2'', z', x_1'', t, s_1, s_0\}$ is a $B_{2,3}$, a contradiction.

Now we assume that C is not joined to C'. Let R be a shortest path from x_1'' to y_1'' with all internal vertices in D_1''. Then the subgraph induced by $\{x_2'', t, s_1, s_0\} \cup V(R) \cup \{y_2''\}$ is a $B_{2,l}$ with $l \geq 3$, a contradiction. □

We complete the proof of this case by reaching our final contradiction, as follows.

Let $N_1 = \{x_i, x_i' : 1 \leq i \leq k\}$ such that for every i with $1 \leq i \leq k$, $\{x_i, x_i'\}$ is a cut. Let H_i be the component of $G - \{x_i, x_i'\}$ not containing v, and let R_i be a perfect path of H_i to $\{x_i, x_i'\}$. Then $R = s_1x_1R_1x_1' \cdots x_kR_kx_k's_2$ is a Hamilton path of B_1 from s_1 to s_2, our final contradiction.

7.7 Proof of Theorem 7.7

Let G be a 2-connected $\{K_{1,3}, N_{1,1,3}\}$-free graph. Adopting the notation and set-up of Section 7.4 we are going to prove that G has a Hamilton path starting from a vertex v, in case $G-r$ contains a nontrivial inner block B_q and all other inner and end blocks of $G - r$ are trivial. Recall that it is sufficient to prove that B_q contains a Hamilton path from s_q to s_{q+1}. Suppose to the contrary that there is no such path. Set

$$N_i = \{v \in B_q - s_{q+1} : d_{B_q - s_{q+1}}(v, s_q) = i\}, \text{ and } j = \max\{i : N_i \neq \emptyset\}.$$

Note that $N_0 = \{s_q\}$ and $N_1 = N_{B_q}(s_q) \setminus \{s_{q+1}\}$.

We already know from Observation 1 that $N_{B_q}(s_q)$ is a clique and $N_{B_q}(s_{q+1})$ is a clique. In particular, this implies that N_1 is a clique. There is nothing to prove if $N_2 = \emptyset$, so we assume $N_2 \neq \emptyset$. We complete the proof of this case by first proving a number of claims.

Claim 1. $rs_q \in E(G)$; $rs_{q+1} \in E(G)$.

Proof. Suppose that $rs_q \notin E(G)$. Let Q be a shortest path from s_q to s_{p+1} containing rs_{p+1} and with all internal vertices outside B_q. Then Q is an induced path with origin s_q and internal vertices outside B_q and of length at least 3.

Note that N_1 is a clique. We first show that all N_i are cliques.

Claim 1.1. For every i with $1 \leq i \leq j$, N_i is a clique.

Proof. We use induction on i. The result is true for $i = 1$. Thus we assume that $2 \leq i \leq j$.

Let x and x' be two vertices in N_i. Suppose $xx' \notin E(G)$. Then x and x' have no common neighbors in N_{i-1}. Let w be a neighbor of x in N_{i-1}, and let w' be a neighbor of x' in N_{i-1}. By the induction hypothesis, $ww' \in E(G)$. Let v be a neighbor of w in N_{i-2}. Then $w'v \in E(G)$; otherwise the subgraph induced by $\{w, v, w', x\}$ is a claw. Let R be a shortest path of $B_q - s_{q+1}$ from v to s_q. Then the subgraph induced by $\{w, x, w', x'\} \cup V(R) \cup V(Q)$ is an $N_{1,1,\ell}$ with $\ell \geq 3$, a contradiction. Thus $xx' \in E(G)$, completing the proof. □

Using the above observations and Lemma 2, we conclude that B_q contains a Hamilton path from s_q to s_{q+1}, a contradiction. Hence we get that $rs_q \in E(G)$. The second assertion follows by symmetry. □

We note here that in the above argumentation we have implicitly proved Lemma 3 in case $F = N_{1,1,3}$.

By Claim 1, $rs_q, rs_{q+1} \in E(G)$. If $p \geq 3$, G contains a claw centered at r, a contradiction. So $p = 2$, $q = 1$, and $G - r$ consists of three blocks. Recall that the two end blocks B_0 and B_2 are both trivial, so rs_0s_1r and rs_2s_3r are two triangles. We again obtain more information on the structure of N_i by proving the following claims.

Claim 2. $j \leq 3$, and if $s_1s_2 \in E(G)$, then N_3 is P_3-free.

Proof. We first deduce that N_2 is not a clique by showing the following.

Claim 2.1. If N_2 is a clique, then for every i with $2 \leq i \leq j$, N_i is a clique.

Proof. Let $Q = s_1rs_3$. Then Q is an induced path with origin s_1 and internal vertices outside B_1 and of length 2.

For $i = 2$, the assertion is true by our assumption. So let $i \geq 3$, and let x and x' be two vertices in N_i. If $xx' \notin E(G)$, then x and x' have no common neighbors in N_{i-1}. Let w be a neighbor of x in N_{i-1}, and let w' be a neighbor of x' in N_{i-1}. By the induction hypothesis, $ww' \in E(G)$. Let v be a neighbor of w in N_{i-2}. Then $w'v \in E(G)$; otherwise the subgraph induced by $\{w, v, w', x\}$ is a claw. Let R be a shortest path of $B_1 - s_2$ from v to s_1. Then the subgraph induced by $\{w, x, w', x'\} \cup V(R) \cup V(Q)$ is an $N_{1,1,\ell}$ with $\ell \geq 3$, a contradiction. Thus $xx' \in E(G)$, completing the proof. □

If for every i with $1 \leq i \leq j$, N_i is a clique, then Lemma 2 implies that B_1 contains a Hamilton path from s_1 to s_2, a contradiction. So we assume that N_2 is not a clique.

Next suppose $j \geq 4$. Let z be a vertex in N_4, let y be a neighbor of z in N_3, and let x be a neighbor of y in N_2. Let x' be a vertex in N_2 other than x. We claim that $xx' \in E(G)$. Assume the contrary. Then $x'y \notin E(G)$; otherwise the subgraph induced by $\{y, x, x', z\}$ is a claw. Besides, x and x' have no common neighbors in N_1. Let w be a neighbor of x in N_1, and let w' be a neighbor of x' in N_1. Then $wx', w'x \notin E(G)$, and the subgraph induced by $\{s_1, s_0, w', x', w, x, y, z\}$ is an $N_{1,1,3}$, a contradiction. This implies that x is adjacent to all other vertices in N_2. Now letting x' and x'' be two vertices in N_2 other than x, we claim that $x'x'' \in E(G)$. Assume the contrary. If $x'y \in E(G)$, then similarly as before, x' is adjacent to all the other vertices in N_1, and then $x'x'' \in E(G)$. Thus we assume that $x'y \notin E(G)$, and similarly $x''y \notin E(G)$. Then the subgraph induced by $\{x, x', x'', y\}$ is a claw, a contradiction. This implies that N_2 is a clique, a contradiction. Thus we get that $j \leq 3$.

Suppose now that $s_1s_2 \in E(G)$, and that $yy'y''$ is an induced P_3 in N_3. Let x be a neighbor of y' in N_2. Then either xy or $xy'' \notin E(G)$. Without loss of generality, we assume that $xy'' \notin E(G)$. Let w be a neighbor of x in N_1. Then $s_2w \in E(G)$; otherwise the subgraph induced by $\{s_1, s_0, s_2, w\}$ is a claw. Now the subgraph induced by $\{s_1, s_0, s_2, s_3, w, x, y', y''\}$ is an $N_{1,1,3}$, a contradiction. □

We next show that s_1 and s_2 are neighbors in B_1.

Claim 3. $s_1s_2 \in E(G)$.

Proof. Assume the contrary. Let d be the distance between s_1 and s_2 in B_1, and let Q be a shortest path from s_1 to s_2 in B_1. Then $d \geq 2$ and, since $j \leq 3$, we have $d \leq 4$. We distinguish three cases according to the value of d.

Case A. $d = 2$.

Noting that we have not used $B_{1,4}$-freeness in Case A of the proof of Claim 3 in Section 7.5, this case can be proved completely analogously.

Case B. $d = 3$.

Let $Q = s_1xys_2$. Similarly as in Case B of the proof of Claim 3 in Section 7.5, we can prove that $\{x, y\}$ is a cut of G. Note that $N_{B_1}(s_1)$ and $N_{B_1}(s_2)$ are disjoint; otherwise $d = 2$. Let H be the component of $G - \{x, y\}$ not containing r. Let x' be a vertex in $N_{B_1}(s_1)$ other than x, and let y' be a vertex in $N_{B_1}(s_2)$ other than y.

If there is a vertex in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$, then without loss of generality, we assume that z is such a vertex and $zx' \in E(G)$. Then the subgraph induced by $\{s_1, s_0, x', z, x, y, s_2, s_3\}$ is an $N_{1,1,3}$, a contradiction. Thus we assume that there are no vertices in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$.

If H contains a vertex nonadjacent with x, then let z' be a vertex with distance 2 from x in H, and let z be a common neighbor of x and z' in H. Then $yz \in E(G)$; otherwise the subgraph induced by $\{x, s_1, y, z\}$ is a claw. $yz' \notin E(G)$; otherwise the subgraph induced by $\{y, x, z', s_2\}$ is a claw. Now the subgraph induced by $\{y, y', z, z', x, s_1, r, s_3\}$ is an $N_{1,1,3}$, a contradiction. Thus we assume that every vertex in H is adjacent to x. Then by Lemma 1, H is a clique.

Let R' be a Hamilton path of $H \cup \{x, y\}$ from x to y, let T be a Hamilton path of $N_{B_1}(s_1)$ from x to x', and let T' be a Hamilton path of $N_{B_1}(s_2)$ from y to y'. Then $R = s_1x'TxR'yT'y's_2$ is a Hamilton path of B_1 from s_1 to s_2, a contradiction.

Case C. $d = 4$.

Let $Q = s_1xyzs_2$. Similarly as in Case C of the proof of Claim 3 in Section 7.5, we can prove that $\{x, z\}$ is a cut of G. Let x' be a vertex in $N_{B_1}(s_1)$ other than x. Note that $N_{B_1}(s_1)$ and $N_{B_1}(s_2)$ are disjoint and not adjacent; otherwise $d \leq 3$. There must be some vertex in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$; otherwise $\{r, x\}$ is a cut. Without loss of generality, we assume that y' is such a vertex, and $x'y' \in E(G)$. Then the subgraph induced by $\{s_1, s_0, x', y', x, y, z, s_2\}$ is an $N_{1,1,3}$, a contradiction. □

By Observation 1 and Claim 3, $N_{B_1}(s_2) \setminus \{s_1\} = N_{B_1}(s_1) \setminus \{s_2\} = N_1$, and by Claims 2 and 3, N_3 is P_3-free. Our next observation shows that N_1 can be partitioned into cut pairs.

Claim 4. For every vertex $x \in N_1$, there is unique vertex $x' \in N_1 \setminus \{x\}$ such that $\{x, x'\}$ is a cut.

Proof. Assume the contrary. Similarly as in the proof of Claim 4 in Section 7.5, there is a vertex $y \in \bigcup_{i=2}^{j} N_i$ such that $\{x, y\}$ is a cut. Let H be the component of $G - \{x, y\}$ not containing r, and let R be a shortest path from x to y with all internal vertices in H.

Let R' be a shortest path in $G - x$ from y to N_1, and let x' be the end vertex of R' other than y. Similarly as in Section 7.5, x' is contained in a cut $\{x', y'\}$ with the other vertex $y' \in V(R') \setminus \{x', y\}$. Let T' be the subpath of R' from y to y', let H' be the component of $G - \{x', y'\}$ not containing r, and let z' be a neighbor of y' in H'. Then the subgraph induced by $\{s_1, s_0, s_2, s_3\} \cup V(R) \cup V(T') \cup \{z'\}$ is an $N_{1,1,\ell}$ with $\ell \geq 3$, a contradiction. Thus there is a vertex $x' \in N_1$ such that $\{x, x'\}$ is a cut.

One can prove the uniqueness similarly as in the proof of Claim 4 in Section 7.5. □

By Claim 4, we can partition N_1 into pairs such that each pair is a cut. These pairs have a nice property with respect to perfect paths, as follows.

Claim 5. Let $\{t, t'\}$ be a cut of G such that $t, t' \in N_1$, and let H be the component of $G - \{t, t'\}$ not containing r. Then there is a perfect path of H to $\{t, t'\}$.

Proof. If $H \cap N_2$ contains only one vertex x, then by the 2-connectedness of G, $H \cap N_3 = \emptyset$ and $xt, xt' \in E(G)$. Then $R = txt'$ is a perfect path of H to $\{t, t'\}$. Thus we assume that $H \cap N_2$ contains at least two vertices. Note that both t and t' are adjacent to some vertices in $H \cap N_2$. We can divide $H \cap N_2$ into two nonempty subset C and C' such that every vertex of C is adjacent to t and every vertex of C' is adjacent to t'.

Recall that $j \leq 3$ and that N_3 is P_3-free, so every component of $H \cap N_3$ is a clique. The proof of the next observations is completely analogous to the proof of Claim 5.1 in Section 7.5.

Claim 5.1. Let D be a component of $H \cap N_3$. If D is joined to C but not to C', then D supports a perfect path to C; if D is joined to C' but not to C, then D supports a perfect path to C'; and if D is joined to both C and C', then D contains a perfect path to C and C'.

We proceed similarly as in Section 7.5.

Let $\mathcal{D} = \{D_1, D_2, \ldots, D_k\}$ be the set of components in $H \cap N_3$ that are joined to C but not to C', let R_i $(1 \leq i \leq k)$ be a perfect path of D_i to C, and let x_i, y_i be the two end vertices of R_i; let $\mathcal{D}' = \{D'_1, D'_2, \ldots, D'_{k'}\}$ be the set of components in $H \cap N_3$ that are joined to C' but not to C, let R'_i $(1 \leq i \leq k')$ be a perfect path of D'_i to C', and let x'_i, y'_i be the two end vertices of R'_i; let $\mathcal{D}'' = \{D''_1, D''_2, \ldots, D''_{k''}\}$ be the set of components in $H \cap N_3$ that are joined to both C and C', let R''_i $(1 \leq i \leq k'')$ be a perfect path of D''_i to C and C', and let x''_i, y''_i be the two end vertices of R''_i, where $x''_i \in C$ and $y''_i \in C'$.

If k'' is odd, or k'' is even and there is a good edge joining C to C', then we can prove the assertion similarly as in Section 7.5. Thus we assume that k'' is even and that every edge joining C to C' is not good. Similarly as in Section 7.5, note that $k'' \geq 2$.

Let R be a shortest path from x''_1 to y''_1 with all internal vertices in D''_1. Then the subgraph induced by $\{s_1, s_0, s_2, s_3, t\} \cup V(R) \cup \{y''_2\}$ is an $N_{1,1,\ell}$ with $\ell \geq 3$, a contradiction. □

We complete the proof of this case by reaching our final contradiction, as follows.

Let $N_1 = \{x_i, x'_i : 1 \leq i \leq k\}$ such that for every i with $1 \leq i \leq k$, $\{x_i, x'_i\}$ is a cut. Let H_i be the component of $G - \{x_i, x'_i\}$ not containing r, and let R_i be the perfect path of H_i to $\{x_i, x'_i\}$. Then $R = s_1 x_1 R_1 x'_1 \cdots x_k R_k x'_k s_2$ is a Hamilton path of B_1 from s_1 to s_2, our final contradiction.

7.8 Remarks

In this section we consider heavy subgraph conditions for a 2-connected graph to be homogeneously traceable.

If a graph is P_3-heavy, then it is hamiltonian (see Chapter 6), and thus homogeneously traceable. As before, it is not hard to see that the statement 'every 2-connected S-heavy graph is homogeneously traceable' only holds when $S = P_3$. Now we consider pairs of graphs $\{R, S\}$ such that every 2-connected $\{R, S\}$-heavy graph is homogeneously traceable.

First note that for each pair $\{K_{1,3}, S\}$, with $S = P_4$, P_5, C_3, Z_1, Z_2, B, N or W, every 2-connected $\{K_{1,3}, S\}$-heavy graph is homogeneously traceable. This can be deduced from the following theorem (see Chapter 6) and the fact that every hamiltonian graph is homogeneously traceable.

Theorem 7.8. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, C_3, Z_1, Z_2, B, N or W.*

In fact, as we will show below, these are the only pairs with this property.

Theorem 7.9. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-heavy implies G is homogeneously traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, C_3, Z_1, Z_2, B, N or W.*

Proof. The 'if' part of the theorem can be obtained from Theorem 7.8 immediately. Now we prove the 'only-if' part of the theorem.

We first construct several graphs that are not homogeneously traceable and are sketched in Figure 7.3.

Let R, S be the two connected graph from the statement in Theorem 7.9. From Theorem 7.4 we can deduce that (up to symmetry) $R = K_{1,3}$, and S is an induced subgraph of $B_{1,4}$, $B_{2,3}$ or $N_{1,1,3}$.

Note that the graph G_1 of Figure 7.3 is $\{K_{1,3}, P_6\}$-heavy, G_2 is $\{K_{1,3}, Z_3\}$-heavy and G_3 is $\{K_{1,3}, N_{1,1,2}\}$-heavy, but that these graphs are not homogeneously traceable. Thus we conclude that S is not an induced supergraph of P_6, Z_3, or $N_{1,1,2}$. This implies $S = P_4$, P_5, C_3, Z_1, Z_2, B, N or W. □

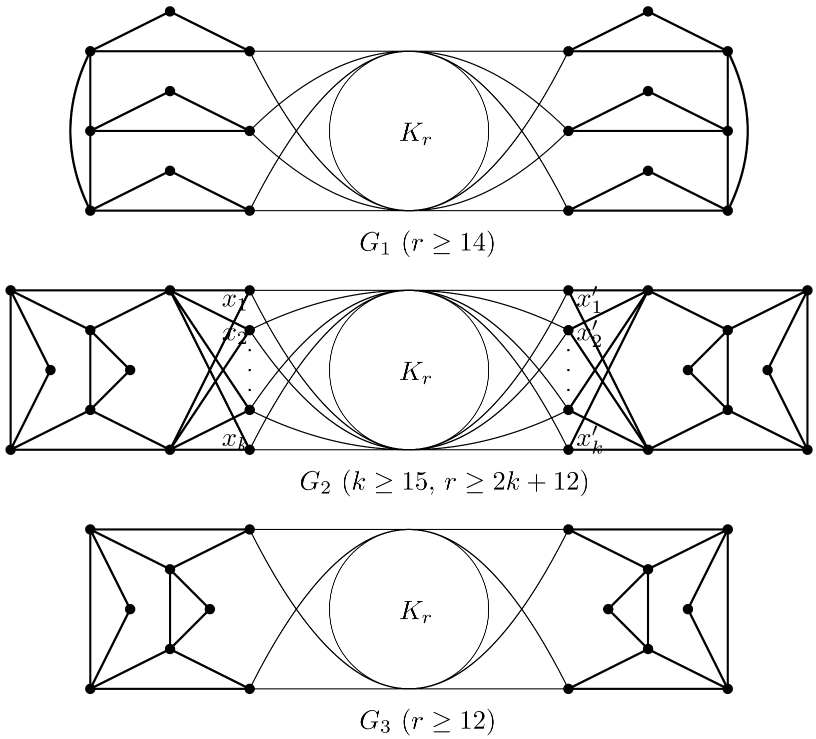


Figure 7.3: Some graphs that are not homogeneously traceable (II)

Chapter 8

Heavy pairs for pancyclicity

8.1 Introduction

A graph G on n vertices is said to be *hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle passing through all the vertices of G. If G contains cycles of length k for every k with $3 \leq k \leq n$, we say that G is *pancyclic*. Note that a pancyclic graph is necessarily hamiltonian. Bedrossian [3] studied forbidden subgraph conditions for a 2-connected graph to be hamiltonian and to be pancyclic. In his PhD thesis, he proved the following nice results, characterizing all pairs of forbidden subgraphs for these properties. We note that a connected P_3-free graph is a complete graph, and hence it is hamiltonian and pancyclic if it has order at least 3. In fact, it is not hard to show that the statement 'every connected H-free graph is hamiltonian (pancyclic)' only holds if $H = P_3$. The case with pairs of forbidden subgraphs (different from P_3) is much more interesting.

Theorem 8.1 (Bedrossian [3]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, P_6, C_3, Z_1, Z_2, B, N or W.*

Theorem 8.2 (Bedrossian [3]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, Z_1 or Z_2.*

Forbidding pairs of graphs as induced subgraphs might impose such a strong condition on the graphs under consideration that hamiltonian properties are almost trivially obtained. As an example, one easily shows that, apart from paths and cycles, connected $\{K_{1,3}, Z_1\}$-free graphs are only a matching away from complete graphs, i.e., their complements consist of isolated vertices and isolated edges. This is one of the motivations to relax forbidden subgraph conditions to conditions in which the subgraphs are allowed, but where additional conditions are imposed on these subgraphs if they appear. Early examples of this approach in the context of hamiltonicity and pancyclicity date back to the early 1990s [4,12]. The idea to put a minimum degree bound on one or two of the end vertices of an induced claw has been explored in [11]. Here we follow the ideas and terminology of [17] by putting an Ore-type degree sum condition on at least one pair of nonadjacent vertices in certain induced subgraphs. These degree sum conditions refer to one of the earliest papers in this area, in which Ore [30] proved that a graph G on $n \geq 3$ vertices is hamiltonian if the degree sum of any two nonadjacent vertices of G is at least n.

The counterpart of Theorem 8.1 for heavy subgraphs was studied in Chapter 6. For hamiltonicity of 2-connected graphs we obtained the following result (see Chapter 6).

Theorem 8.3. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, C_3, Z_1, Z_2, B, N or W.*

Is there a natural counterpart of Theorem 8.2 involving heavy subgraphs? What can we say about the pancyclicity of graphs when we consider heavy subgraph conditions instead of forbidden subgraph conditions? To start with a negative observation, let us first consider the complete bipartite graph $K_{n/2,n/2}$. Note that every induced subgraph of $K_{n/2,n/2}$ (other than P_1 and P_2) is heavy, but $K_{n/2,n/2}$ is clearly not pancyclic. This implies that for any family $\mathcal{H}$ of graphs, a 2-connected graph G (not being a cycle) cannot be guaranteed to be pancyclic by imposing that G is $\mathcal{H}$-heavy. As in existing degree condition results for pancyclicity, we have to impose a slightly stronger degree condition in order to exclude the above counterexamples.

Imposing a natural slightly stronger Ore-type degree condition, we obtain the following counterpart of Theorem 8.2.

Theorem 8.4. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-o_1-heavy implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, Z_1 or Z_2.*

The 'only if' part of the theorem follows almost directly from Theorem 8.2, since H-free graphs are H-o_1-heavy. For the 'if' part of the theorem, noting that P_4 and Z_1 are both induced subgraphs of Z_2, it is sufficient to prove the following result.

Theorem 8.5. *Let G be a 2-connected graph which is not a cycle. If G is $\{K_{1,3}, P_5\}$-o_1-heavy or $\{K_{1,3}, Z_2\}$-o_1-heavy, then G is pancyclic.*

The proof is by induction on the order n of the graph. Using Theorem 8.2, we are done if G is $\{K_{1,3}, P_5\}$-free or $\{K_{1,3}, Z_2\}$-free, so we may assume there is at least one pair of vertices with degree sum at least $n + 1$. Cycles of length 3, 4 and 5 are easily obtained from the degree conditions, and cycles of length n and $n - 1$ are easily obtained by using Theorem 8.3 directly, and after establishing the existence of a vertex v whose removal does not affect the 2-connectedness and then using Theorem 8.3 on $G - v$. For the other cases, we are done if we can find a vertex or a pair of vertices whose removal does not affect the 2-connectedness and the degree sum conditions on remaining pairs in the smaller graph. Assuming that such vertices or pairs of vertices do not exist, forces a lot of structure on the vertex cuts S with $|S| = 2$ of G and on the components of $G - S$, enabling us to prove Theorem 8.5. We postpone the details of the proof of Theorem 8.5 to Section 8.3.

Let $F = P_5$ or $F = Z_2$. As we pointed out before, $K_{n/2,n/2}$ is a 2-connected $\{K_{1,3}, F\}$-heavy graph which is not pancyclic. Another graph with this property is $K_{n/2,n/2} - e$ (the graph obtained from $K_{n/2,n/2}$ by deleting an arbitrary edge). Apart from the cycles and these two types of graphs, we do not know whether there exist any other graphs with the above properties, so we raise it as an open problem.

Problem 8.1. Is there some graph G on n vertices other than C_n, $K_{n/2,n/2}$ and $K_{n/2,n/2} - e$ such that G is $\{K_{1,3}, P_5\}$-heavy or $\{K_{1,3}, Z_2\}$-heavy but not pancyclic?

8.2 Some preliminaries

In the next section we will prove Theorem 8.5. Before we do so, in this section we introduce some additional terminology, and we will prove some useful lemmas.

Let G be a graph. For a subgraph H of G, when no confusion can arise we also use H to denote the vertex set of H; and similarly, for a subset S of $V(G)$, we also use S to denote the subgraph of G induced by S.

The following useful lemma is an easy exercise that can be found in [8], but we do not know the precise origin of the result. We present it here without a proof. A stronger result appeared in [5].

Lemma 1. *Let G be a graph on n vertices, and let x be a vertex of G. If $d(x) \geq n/2$ and $G - x$ is hamiltonian, then G is pancyclic.*

Our next result is a structural lemma on claw-o_1-heavy graphs.

Lemma 2. *Let G be a claw-o_1-heavy graph on $n \geq 4$ vertices, and let x and x' be two vertices of G. Then*
(1) *if $xx' \in E(G)$ and $d(x) + d(x') \geq n + 1$, then xx' is contained in a triangle;*
(2) *if $d(x) \geq (n + 1)/2$, then x is contained in a triangle; and*
(3) *if $xx' \notin E(G)$ and $d(x) + d(x') \geq n + 1$, then*
 (a) *x and x' have at least three common neighbors in G, and*
 (b) *x and x' are contained in a common quadrangle and a common pentagon.*

Proof. (1) Since $d(x) + d(x') \geq n + 1$, x and x' have at least one common neighbor y. Then $xyx'x$ is a triangle containing xx'.

(2) Since $d(x) \geq (n + 1)/2$ and $n \geq 4$, $d(x) \geq 3$. Let y, y', y'' be three neighbors of x. If $yy' \in E(G)$, then $xyy'x$ is a triangle containing x. Next assume that $yy' \notin E(G)$, and similarly, $yy'', y'y'' \notin E(G)$. Then the subgraph induced by $\{x, y, y', y''\}$ is a claw. Since G is claw-o_1-heavy, there must be a vertex in $\{y, y', y''\}$ with degree at least $(n+1)/2$. Without loss of generality, we assume that $d(y) \geq (n + 1)/2$. Then $d(x) + d(y) \geq n + 1$. By (1), xy is contained in a triangle.

(3) Here we assume $xx' \notin E(G)$ and $d(x) + d(x') \geq n + 1$. If x and x' have at most two common neighbors, then $d(x) + d(x') \leq (n - 2) + 2 = n$, a

contradiction. Thus x and x' have at least three common neighbors. We may assume without loss of generality that $d(x) \geq (n+1)/2$.

Let y, y', y'' be three common neighbors of x and x'. Then $xyx'y'x$ is a quadrangle containing x and x'. If $yy' \in E(G)$, then $xyy'x'y''x$ is a pentagon containing x and x'. Next assume that $yy' \notin E(G)$, and similarly, $yy'', y'y'' \notin E(G)$. Then the subgraph induced by $\{x, y, y', y''\}$ is a claw. Without loss of generality, we assume that $d(y) \geq (n+1)/2$. Then $d(x) + d(y) \geq n+1$. By (1), xy is contained in a triangle $xyzx$. Noting that $z \neq x', y'$, $xzyx'y'x$ is a pentagon containing x and x'. □

Let G be a graph on n vertices. In the following, we call a vertex x a *super heavy vertex* of G if $d(x) \geq (n+1)/2$, and we call a pair of vertices $\{x, y\}$ a *super heavy pair* of G if $xy \notin E(G)$ and $d(x) + d(y) \geq n+1$. Note that a super heavy pair contains at least one super heavy vertex. The importance of the existence of super heavy vertices for pancyclicity is already demonstrated by Lemma 1. The next lemma relates the (non)existence of such vertices to the structure of the neighborhood of a vertex cut.

Lemma 3. *Let G be a 2-connected claw-o_1-heavy graph, and suppose $\{r, s\}$ is a vertex cut of G. Then*

(1) *$G - \{r, s\}$ has exactly two components; and*

(2) *for any distinct neighbors x and x' of r: x and x' are in a common component of $G - \{r, s\}$ if and only if $xx' \in E(G)$ or $\{x, x'\}$ is a super heavy pair of G.*

Proof. (1) If there are at least three components of $G - \{r, s\}$, then let H, H' and H'' be three such components. Let x, x' and x'' be neighbors of r in H, H' and H'', respectively. Then the subgraph induced by $\{r, x, x', x''\}$ is a claw. Since x and x' have at most the two common neighbors r and s, by Lemma 2, $d(x) + d(x') \leq n$. Similarly, $d(x) + d(x'') \leq n$ and $d(x') + d(x'') \leq n$, contradicting that G is claw-o_1-heavy. Thus, $G - \{r, s\}$ has exactly two components.

(2) If x and x' are not in a common component, then clearly $xx' \notin E(G)$, and since x and x' have at most the two common neighbors r and s, by Lemma 2, $d(x) + d(x') \leq n$. Thus $\{x, x'\}$ is not a super heavy pair. On the other hand, if x and x' are in a common component, then let x'' be a neighbor of r in the component not containing x and x'. If $xx' \notin E(G)$, then the subgraph induced by $\{r, x, x', x''\}$ is a claw and $d(x) + d(x'') \leq n$, $d(x') + d(x'') \leq n$. Since G is

claw-o_1-heavy, $d(x)+d(x') \geq n+1$, so $\{x, x'\}$ is a super heavy pair, completing the proof of Lemma 3. □

In the sequel, by the concept *cut* we always refer to a vertex cut with 2 vertices. A pair of vertices $\{x, y\}$ is called a *separable pair* of G if x and y are in distinct components of $G - \{r, s\}$ for some cut $\{r, s\}$ of G. So by Lemma 3, a separable pair cannot be a super heavy pair.

Let G be a 2-connected graph, let $\{r, s\}$ be a cut of G, and let H be a component of $G - \{r, s\}$. We call the subgraph induced by $H \cup \{r, s\}$ a *link* of G (this is called an $\{r, s\}$-component in [8]). For such a link, $\{r, s\}$ is called the *bolt* of the link, H is called the *inside*, and $H' = G - \{r, s\} - H$ is called the *outside* of the link. Let L be a link of G with bolt $\{r, s\}$ and inside H. Then if its outside $H' = G - \{r, s\} - H$ is connected, then the subgraph induced by $H' \cup \{r, s\}$ is also a link, called the *co-link* of L, and denoted by L_c.

Note that if a link L has a co-link, then its co-link is unique, and L is the co-link of its co-link. By Lemma 3, we see that if a graph G has connectivity 2 and is claw-o_1-heavy, then every link of G has a co-link. It is convenient to denote a link L with bolt $\{r, s\}$ by $L(r, s)$, and its co-link by $L_c(r, s)$.

The next series of lemmas provides some structural information on cuts and links.

Lemma 4. *Let G be a 2-connected graph, let $L(r, s)$ be a link of G, and let H be the inside of L. If $\{r', s'\}$ is a cut of G with $r', s' \in L$, then there is a component of $G - \{r', s'\}$ contained in H.*

Proof. If $\{r', s'\} = \{r, s\}$, then the result is trivially true. So we assume that $\{r', s'\} \neq \{r, s\}$. Without loss of generality, we assume that $r \neq r', s'$. Note that r has a neighbor in every component of $G - \{r, s\}$. Since $r', s' \in H$, every component of $G - \{r, s\}$ other than H is contained in a component of $G - \{r', s'\}$ common to r (and also common to s if $s \neq r', s'$). Thus any other component of $G - \{r', s'\}$ is contained in H. □

Let L be a link of a graph. A vertex of the inside (outside) of L is called a vertex inside (outside) L.

Lemma 5. *Let G be a 2-connected graph, let $L = L(r, s)$ be a link of G, and let x be a vertex inside L. If $\{x, y\}$ is a cut of G for some vertex y outside L, then*

(1) *$rs \notin E(G)$, and r and s are in distinct components of $G - \{x, y\}$;*

(2) *x is a cut vertex of L, and r and s are in distinct components of $L - x$; and*

(3) *$rx \in E(G)$ and x is the only neighbor of r in H, or $\{r, x\}$ is a cut.*

Proof. (1) Let u be an arbitrary vertex inside L other than x. By the 2-connectedness of G, there is a path from u to r or s not passing through x with all internal vertices inside L. Similarly, for an arbitrary vertex v outside L other than y, there is a path from v to r or s not passing through y with all internal vertices outside L. Thus if $rs \in E(G)$ or if r and s are in a common component of $G - \{x, y\}$, then $G - \{x, y\}$ is connected, a contradiction.

(2) Let P be an arbitrary path of L from r to s. Note that P cannot pass through y. By (1), P passes through x. This implies that x is a cut vertex of L, and that r and s are in distinct components of $L - x$.

(3) Suppose that r has a neighbor r' in L other than x. By (1), r and s are in distinct components of $G - \{x, y\}$. Clearly r and r' are in a common component. Let P be an arbitrary path of G from r' to s. Thus P either passes through x or passes through y. If P passes through y, then it also passes through r. This implies that $\{r, x\}$ is a cut. □

Lemma 6. *Let G be a 2-connected claw-o_1-heavy graph, let $L = L(r, s)$ be a link of G, let H be the inside of L, and let x be a vertex in H. Then the following two statements are equivalent:*

(1) *$rx \in E(G)$ and x is the only neighbor of r in H, or $\{r, x\}$ is a cut.*

(1) *$sx \in E(G)$ and x is the only neighbor of s in H, or $\{s, x\}$ is a cut.*

Proof. First assume that $rx \in E(G)$ and x is the only neighbor of r in H. If $sx \notin E(G)$ or s has at least two neighbors in H, then there is a neighbor $s' \neq x$ of s in H. Let P be an arbitrary path of G from s' to r. If P does not pass through s, then every internal vertex of P is in H. Noting that r has only one neighbor x in H, this implies that P then passes through x. Hence $\{s, x\}$ is a cut.

Suppose now that $\{r, x\}$ is a cut. Let H_c be the outside of L. Using Lemma 4, let H' be the component of $G - \{r, x\}$ contained in H. If $sx \notin E(G)$ or s has at least two neighbors in H, then let r' be a neighbor of r in H', let r'_c be a neighbor of r outside L, and let s' be a neighbor of s inside L other than x. Clearly $s' \notin H'$.

We claim that every neighbor of r is either in $H' \cup \{x\}$ or in $H_c \cup \{s\}$. Otherwise, let r'' be a neighbor of r in $H - x - H'$. Then the subgraph induced by $\{r, r', r'_c, r''\}$ is a claw. It is easily seen that any pair of vertices from $\{r', r'_c, r''\}$ is separable. By Lemma 3, the claw induced by $\{r, r', r'_c, r''\}$ is not o_1-heavy, a contradiction.

Recall that r' and s' are in distinct components of $G - \{r, x\}$. Let P be an arbitrary path of G from r' to s'. Then P passes through either r or x. Also recall that every neighbor of r not in $H' \cup \{x\}$ is in $H_c \cup \{s\}$. Thus if P passes through r, then it will also pass through s. This implies that $\{s, x\}$ is a cut. This completes the proof of one direction of the lemma.

The opposite direction follows by symmetry. □

A link $L = L(r, s)$ is said to be *simple* if both r and s have at least two neighbors inside L, and for every vertex x inside L, $\{r, x\}$ and $\{s, x\}$ are not cuts. By Lemma 6, we can see that if $L = L(r, s)$ is a link of a 2-connected claw-o_1-heavy graph, then L is simple if and only if r has at least two neighbors inside L, and for every vertex x inside L, $\{r, x\}$ is not a cut.

Lemma 7. *Let G be a 2-connected claw-o_1-heavy graph, let $L = L(r, s)$ be a link of G, and let H be the inside of L. Then L is 2-connected if and only if $rs \in E(G)$ or L is simple.*

Proof. First we assume that L has a cut vertex x. Clearly r and s are in distinct components of $L - x$; otherwise x is a cut vertex of G. Thus $rs \notin E(G)$. Moreover, if r has at least two neighbors in H, then let r' be a neighbor of r in H other than x. Let P be an arbitrary path of G from r' to s. If P does not pass through r, then every internal vertex of P is in H. Note that x is a cut vertex of L, and clearly r' and r are in a common component of $L - x$. P will pass through x. This implies that $\{r, x\}$ is a cut and L is not simple.

Suppose now that L is 2-connected. We assume that $rs \notin E(G)$. If r has only one neighbor x in H, then clearly x is a cut vertex of L. So we assume that r has at least two neighbors in H. If $\{r, x\}$ is a cut of G for some x in H, then let H' be the component of $G - \{r, x\}$ contained in H, and let H_c be the outside of L. Let P be an arbitrary path of L from r to s. Similarly as in the proof of Lemma 6, we can prove that every neighbor of r is either in $H' \cup \{x\}$ or in H_c. Note that every internal vertex of P is in H. P must pass through x. This implies x is a cut vertex of G, a contradiction. So L is simple. □

Let G be a 2-connected claw-o_1-heavy graph, and let $rxsr$ be a triangle such that $d(x) = 2$, $d(r) \geq 3$, and $d(s) \geq 3$. Then by Lemma 7, we get that $G - x$ is 2-connected. Similarly, let $rxysr$ be a quadrangle such that $d(x) = d(y) = 2$, $d(r) \geq 3$, and $d(s) \geq 3$. Then $G - \{x, y\}$ is 2-connected.

Note that a simple link is not necessarily a minimal one. Now we prove the following lemma.

Lemma 8. *Let G be a 2-connected claw-o_1-heavy graph, let $L = L(r, s)$ be a simple link of G, and let H be the inside of L. Suppose that there is a link L' contained in H. Then there is a link L'' (possibly equal to L') contained in H and containing L' such that its co-link L''_c is simple.*

Proof. We consider a link L'' contained in H and containing L' with the largest order. Let $\{r'', s''\}$ be the bolt, H'' the inside, and H''_c the outside of L''.

By Lemma 6, for each x in H, $\{r, x\}$ and $\{s, x\}$ are not cuts. If r'' has only one neighbor x in H''_c, then $\{s'', x\}$ is a cut and $x \in H$. Then $H'' \cup \{r''\}$ is the component of $G - \{s'', x\}$ contained in H, and the subgraph induced by $H'' \cup \{r'', s'', x\}$ is a link contained in H and containing L' with larger order than L'', a contradiction. Thus we assume that r'' has at least two neighbors in H''_c, and similarly, s'' has at least two neighbors in H''_c.

If $\{r'', x\}$ is a cut of G for some $x \in H''_c$, then note that $x \neq r, s$, and by Lemma 5, $x \notin H_c \cup \{r, s\}$, where H_c is the outside of L. This implies x is inside H. By Lemma 6, $\{s'', x\}$ is a cut. Let H''' be the component of $G - \{r'', x\}$ contained in H. If H'' is contained in H''', then the subgraph induced by $H''' \cup \{r'', x\}$ is a link contained in H and containing L' with larger order than L'', a contradiction. Thus we assume that H'' is not contained in H'''. Note that every neighbor of r'' is either in $H'' \cup \{s''\}$ or in $H''' \cup \{x\}$. $H'' \cup H''' \cup \{r''\}$ is the component of $G - \{s'', x\}$ contained in H, and the subgraph induced by $H'' \cup H''' \cup \{r'', s'', x\}$ is a link contained in H and containing L' with larger order than L'', a contradiction.

Thus we conclude that L''_c is simple. □

Let G be a 2-connected graph. If $G - x$ is 2-connected for a vertex x of G, then we call x a *c-removable vertex* of G (a removable vertex with respect to the connectivity condition); similarly, if $G - \{x, y\}$ is 2-connected for a pair of vertices $\{x, y\}$ of G, then we call $\{x, y\}$ a *c-removable pair* of G. Note that every vertex of a 3-connected graph is c-removable. Also note that every non-removable vertex of a 2-connected graph is contained in a cut. The existence of c-removable vertices and pairs plays a key role in our induction proof of Theorem 8.5 in the next section. Here we prove a preliminary lemma on c-removable pairs.

Lemma 9. *Let G be a 2-connected graph on at least 5 vertices, and let $L = L(r, s)$ and $L' = L'(r', s')$ be two 2-connected links of G that are internally-disjoint. If x and x' are two c-removable vertices of G inside L and L', respectively, then $\{x, x'\}$ is a c-removable pair of G.*

Proof. Let y be an arbitrary vertex of $G - \{x, x'\}$. We prove that $G' = G - \{x, x', y\}$ is connected.

If y is one of the vertices in $\{r, s, r', s'\}$, then without loss of generality, we assume that $y = r$. Then for every vertex u inside L with $u \neq x$, since x is c-removable and $\{r, x\}$ is not a cut, there is a path P of $G - \{r, x\}$ from u to s. Clearly, P does not pass through x'. This implies that u and s are connected by the path P in G'. Similarly, for every vertex v outside L with $v \neq x'$, since x' is c-removable and $\{r, x'\}$ is not a cut, there is a path Q of $G - \{r, x'\}$ from v to s that does not pass through x. This implies that v and s are connected by the path Q in G'. Thus G' is connected.

Now we assume that y is not a vertex of $\{r, s, r', s'\}$. Without loss of generality, we assume that y is outside L. Then for every vertex u inside L with $u \neq x$, since L is 2-connected, there is a path P of $L - x$ from u to r. This implies that u and r are connected by the path P in G'. In particular, r and s are connected in G'. Besides, for every vertex v outside L with $v \neq x', y$, since x' is c-removable and $\{x', y\}$ is not a cut, there is a path Q of $G - \{r, x'\}$ from v to r or s with all internal vertices outside L. This implies that v and r or s are connected by the path Q in G'. Thus G' is connected. □

Let G be a graph and let x be a vertex of G. If every super heavy pair $\{u, v\}$ of G, with $u, v \in V(G) \setminus \{x\}$, is also a super heavy pair of $G - x$ (in terms of the order of the new graph), then we call x a *d-removable vertex* of G (a removable vertex with respect to the degree condition). Let x, y be two distinct vertices of G. If every super heavy pair $\{u, v\}$ of G, with $u, v \in V(G) \setminus \{x, y\}$, is also a super heavy pair of $G - \{x, y\}$, then we call $\{x, y\}$ a *d-removable pair* of G. For an induction proof the existence of vertices or pairs of vertices that are both c-removable and d-removable is very favorable, as we will see in the next section. We finish this section with the following easy observations on d-removable vertices and pairs.

Lemma 10. *Let G be a graph, and let x, y be two distinct vertices of G. Then*

(1) *if $N(x)$ contains no super heavy pair of G, then x is a d-removable vertex of G; and*

(2) *if x and y have no common neighbors, then $\{x, y\}$ is a d-removable pair of G.*

Proof. We use n to denote the order of G.

(1) Let $G' = G - x$, and let $\{u, v\}$ be an arbitrary super heavy pair of G with $u, v \in V(G) \setminus \{x\}$. If $N(x)$ contains no super heavy pairs of G, then at least one of u and v is not in $N(x)$. Without loss of generality, we assume that $u \notin N(x)$. Then $d_{G'}(u) = d(u)$ and $d_{G'}(v) \geq d(v) - 1$. Thus $d_{G'}(u) + d_{G'}(v) \geq n$. Since the order of G' is $n - 1$, $\{u, v\}$ is a super heavy pair of G'. This implies that x is a d-removable vertex of G.

(2) Let $G' = G - \{x, y\}$, and let $\{u, v\}$ be an arbitrary super heavy pair of G with $u, v \in V(G) \setminus \{x, y\}$. If x and y have no common neighbors, then at least one of ux and uy is not in $E(G)$. Then $d_{G'}(u) \geq d(u) - 1$, and similarly, $d_{G'}(v) \geq d(v) - 1$. Thus $d_{G'}(u) + d_{G'}(v) \geq n - 1$. Since the order of G' is $n - 2$, $\{u, v\}$ is a super heavy pair of G'. This implies that $\{x, y\}$ is a d-removable pair of G. □

8.3 Proof of Theorem 8.5

Let G be a 2-connected $\{K_{1,3}, P_5\}$-o_1-heavy or $\{K_{1,3}, Z_2\}$-o_1-heavy graph that is not a cycle, and let $n = |V(G)|$. We are going to prove that G is a pancyclic graph by induction on n. If G contains only three vertices, then the result is trivially true. So we assume that $n \geq 4$.

If G is $\{K_{1,3}, P_5\}$-free or $\{K_{1,3}, Z_2\}$-free, then by Theorem 8.2, G is pancyclic. So we assume that G is neither $\{K_{1,3}, P_5\}$-free nor $\{K_{1,3}, Z_2\}$-free. This implies that G contains at least one super heavy pair.

By Lemma 2, G contains a triangle, a quadrangle and a pentagon. Next we are going to prove a number of claims. Our first claim establishes the existence of long cycles.

Claim 1. G contains a cycle of length n and a cycle of length $n-1$.

Proof. Since G is $\{K_{1,3}, P_5\}$-o_1-heavy or $\{K_{1,3}, Z_2\}$-o_1-heavy, by Theorem 8.3, G is hamiltonian. So G contains a cycle of length n.

Let C be a Hamilton cycle of G, and let $\{r, s\}$ be a super heavy pair of G. Clearly $\{r, s\}$ divides C into two subpaths. Recall from the definition of a super heavy pair that $rs \notin E(G)$. Let $P = rx_1x_2 \cdots x_ks$ and $Q = ry_1y_2 \cdots y_\ell s$ be the two subpaths of C, where $k+\ell+2 = n$. If $rx_2 \in E(G)$, then $C' = C - rx_1x_2 \cup rx_2$ (with the obvious meaning) is a cycle of length $n-1$. Thus we assume that $rx_2 \notin E(G)$ and, similarly $sx_{k-1}, ry_2, sy_{\ell-1} \notin E(G)$. Since $d(r)+d(s) \geq n+1$, there must be a vertex x_i, $2 \leq i \leq k-1$, such that $rx_{i+1}, sx_{i-1} \in E(G)$, or a vertex y_j, $2 \leq j \leq \ell-1$, such that $ry_{j+1}, sy_{j-1} \in E(G)$. Without loss of generality, we assume that there is a vertex x_i, $2 \leq i \leq k-1$, such that $rx_{i+1}, sx_{i-1} \in E(G)$. Clearly x_i is a c-removable vertex of G.

Let $G' = G - x_i$. Then G' is 2-connected. Let $\{u, v\}$ be an arbitrary super heavy pair of G. Noting that $d_{G'}(u) \geq d(u) - 1$ and $d_{G'}(v) \geq d(v) - 1$, we have $d_{G'}(u)+d_{G'}(v) \geq n-1$. Since G' has $n-1$ vertices, $\{u, v\}$ is a *heavy pair* of G', i.e., u, v are nonadjacent and with degree sum at least $|V(G')|$. This implies that G' is $\{K_{1,3}, P_5\}$-heavy or $\{K_{1,3}, Z_2\}$-heavy. Hence, by Theorem 8.3, G' contains a Hamilton cycle, which is a cycle of length $n-1$. □

By Lemma 2 and Claim 1, if $n \leq 7$, then G is pancyclic. So we assume that $n \geq 8$. It suffices to prove that G contains a cycle of length k for all $k \in [6, n-2]$.

Suppose to the contrary that G does not contain cycles of all these lengths. Our next claim shows that G has no vertices or vertex pairs that are c-removable and d-removable at the same time.

Claim 2. G contains no vertices or pairs that are both c-removable and d-removable.

Proof. If G contains a vertex x that is both c-removable and d-removable, then $G' = G - x$ is 2-connected and $\{K_{1,3}, P_5\}$-o_1-heavy or $\{K_{1,3}, Z_2\}$-o_1-heavy. By the induction hypothesis, G' contains a cycle of length k for all $k \in [3, n-1]$, a contradiction. Similarly, if G contains a pair of vertices $\{x, y\}$ that is both c-removable and d-removable, then $G' = G - \{x, y\}$ is 2-connected and $\{K_{1,3}, P_5\}$-o_1-heavy or $\{K_{1,3}, Z_2\}$-o_1-heavy. By the induction hypothesis, G' contains a cycle of length k for all $k \in [3, n-2]$, a contradiction. □

The next claim shows that super heavy vertices must be part of a cut of G.

Claim 3. Every super heavy vertex of G is contained in a cut.

Proof. Let r be a super heavy vertex of G. If r is not contained in any cut, then r is c-removable and $G-r$ is 2-connected. Similarly as in the proof of Claim 1, we can prove that $G-r$ is $\{K_{1,3}, P_5\}$-heavy or $\{K_{1,3}, Z_2\}$-heavy, and hence hamiltonian. By Lemma 1, G is pancyclic, a contradiction. □

The following claim provides useful structural properties related to the links of G.

Claim 4. Let $L = L(r, s)$ be a link of G, and let H be the inside of L. Then one of the following statements holds.

(1) H contains a c-removable vertex of G, or
(2) L is an induced path from r to s.

Proof. We use induction on $|V(H)|$. If H consists of only one vertex x, then $rx, sx \in E(G)$. If $rs \in E(G)$, then by Lemma 7, $G-x$ is 2-connected, hence x is c-removable and (1) holds. If $rs \notin E(G)$, then L is an induced path rxs and (2) holds. Thus we assume that H has at least two vertices. Suppose that both statements of the claim do not hold. We prove a number of subclaims to reach a contradiction.

Claim 4.1. There is a vertex in H with degree at least 3.

Proof. Suppose that every vertex of H has degree 2. If $rs \notin E(G)$, then L is an induced path and (2) holds. So we assume that $rs \in E(G)$. If H consists of two vertices x_1 and x_2, then by Lemma 7, $G - \{x_1, x_2\}$ is 2-connected, hence $\{x_1, x_2\}$ is a c-removable pair of G. By Lemma 10, $\{x_1, x_2\}$ is a d-removable pair of G, a contradiction to Claim 2. Thus we assume that H has $k \geq 3$ vertices.

Let $rx_1x_2 \cdots x_ks$ be the path from r to s, where $x_i \in H$, $1 \leq i \leq k$. Note that x_i cannot be in a super heavy pair of G since $d(x_i) = 2$. Let y be a neighbor of r outside L and z be a neighbor of s outside L. Then $yrx_1x_2x_3$ is an induced P_5 which is not o_1-heavy. At the same time, if $sy \in E(G)$, then the subgraph induced by $\{r, s, y, x_1, x_2\}$ is a Z_2 which is not o_1-heavy. Thus G will be neither P_5-o_1-heavy nor Z_2-o_1-heavy, a contradiction. So we assume that $sy \notin E(G)$ and similarly, $rz \notin E(G)$. Then the subgraph induced by

$\{r, s, y, x_1\}$ is a claw. Thus $d(s)+d(y) \geq n+1$, and similarly, $d(r)+d(z) \geq n+1$. This implies that $d(r) + d(y) \geq n + 1$ or $d(s) + d(z) \geq n + 1$. Without loss of generality, we assume that $d(r) + d(y) \geq n + 1$. Then by Lemma 2, ry is contained in a triangle $ryy'r$. Now the subgraph induced by $\{y, y', r, x_1, x_2\}$ is a Z_2 which is not o_1-heavy, a contradiction. □

Claim 4.2. L is simple.

Proof. If r has only one neighbor x in H, then s has a neighbor in H other than x; otherwise x would be a cut vertex of G. By Lemma 6, $\{s, x\}$ is a cut. Let H' be the component of $G - \{s, x\}$ contained in H. Then $L' = H' \cup \{s, x\}$ is a link contained in L. Clearly, every vertex in L is either r or in L'. By the induction hypothesis, either H' contains a c-removable vertex of G or L' is an induced path from s to x. If L' is an induced path from s to x, then every vertex in H will have degree 2, a contradiction. Thus H' contains a c-removable vertex of G, and it is also contained in H, a contradiction.

Thus we next assume that r has at least two neighbors in H, and similarly, that s has at least two neighbors in H.

If there is a vertex x in H such that $\{r, x\}$ is a cut, then by Lemma 6, $\{s, x\}$ is a cut. Let H' be the component of $G - \{r, x\}$ contained in H, and let H'' be the component of $G - \{s, x\}$ contained in H. Then $L' = H' \cup \{r, x\}$ and $L'' = H'' \cup \{s, x\}$ are two links contained in L. Clearly, every vertex in L is either in L' or in L''. If both L' and L'' are induced paths, then every vertex in H will have degree 2, a contradiction. Thus we assume that L' or L'' is not an induced path. By the induction hypothesis, H' or H'' contains a c-removable vertex of G, and it is also contained in H, a contradiction. □

Claim 4.3. There is a link contained in H. Moreover, if H contains a super heavy vertex, then there is a link contained in H and containing a super heavy vertex.

Proof. Let x be an arbitrary vertex of H. If x is not c-removable, then x is contained in a cut $\{x, y\}$. By Claim 4.2, $\{r, x\}$ and $\{s, x\}$ are not cuts. Thus $y \neq r$ or $y \neq s$, and by Lemma 5, $y \in H$. This implies that there is a link L' contained in H (and containing x). In particular, if H contains a super heavy vertex x', then there is a link L' contained in H and containing x'. □

Here we continue the proof of Claim 4. By Lemma 8, there is a link contained in H such that its co-link is simple. Moreover, if H contains a super heavy vertex x, then by Claim 4.3 there is a link contained in H and containing x. Then by Lemma 8, there is a link contained in H and containing x such that its co-link is simple. Denote this link by L', let x be a vertex inside L' and assume that x is super heavy if H contains the super heavy vertex x. Let $\{r', s'\}$ be the bolt, H' the inside, and H'_c the outside of L'. If H' contains a c-removable vertex of G, then it is also a c-removable vertex contained in H, a contradiction. So by the induction hypothesis, we assume that L' is an induced path.

If H' consists of only one vertex x, then since $L'_c = G - x$ is simple, by Lemma 7, x is a c-removable vertex of G, and it is also contained in H, a contradiction. If H' consists of only two vertices x_1 and x_2, then since $L'_c = G - \{x_1, x_2\}$ is simple, by Lemma 7, $\{x_1, x_2\}$ is a c-removable pair of G and by Lemma 10, $\{x_1, x_2\}$ is a d-removable pair of G, a contradiction to Claim 2. Thus we assume that H' contains $k \geq 3$ vertices.

Let $r'x_1x_2 \cdots x_ks'$ be the path of L' from r' to s', where $x_i \in H'$, $1 \leq i \leq k$.

Note that x_i cannot be in a super heavy pair of G since $d(x_i) = 2$. Let y be a neighbor of r' in H'_c, and let z be a neighbor of s' in H'_c. Then $yr'x_1x_2x_3$ is an induced P_5 of G which is not o_1-heavy. At the same time, if r' is contained in a triangle, then we assume that $r'yy'r'$ is a triangle. Then the subgraph induced by $\{y, y', r', x_1, x_2\}$ is a Z_2 which is not o_1-heavy, a contradiction. Thus we assume that r' is not contained in a triangle. By Lemma 2, r' is not super heavy. Similarly, we get that s' is not contained in a triangle and is not super heavy. This implies that there are no super heavy vertices in H.

Since L'_c is simple, r' has at least two neighbors in H'_c. Let y' be a neighbor of r' in H'_c other than y. Note that r' is contained in no triangles, $yy' \notin E(G)$, and the subgraph induced by $\{r', y, y', x_1\}$ is a claw. Since $d(x_1) = 2$, either y or y' is a super heavy vertex of G. Without loss of generality, we assume that y is super heavy. Since H contains no super heavy vertex, r' has at most one neighbor in $H - H'$, and $y = r$ or $y = s$. Without loss of generality, we assume that $y = r$. Note that r is a super heavy vertex. By Lemma 2, r is contained in a triangle $rtt'r$.

If $t \in H$, then $\{r', t\}$ is not a super heavy pair, since r' and t are not super heavy vertices. If $t = s$, then $\{r', t\}$ is not a super heavy pair, since r' has at most one neighbor in $H - H'$. If $t \in G - L$, then $\{r', t\}$ is not a super heavy pair by Lemma 3. Similarly, $\{r', t'\}$ is not a super heavy pair of G. Thus the subgraph induced by $\{t, t', r, r', x_1\}$ is a Z_2 which is not o_1-heavy, a contradiction. $\square$

The next claim provides useful information on the existence of c-removable vertices in the inside of a simple link.

Claim 5. Let $L = L(r, s)$ be a simple link of G, and let H be the inside of L. Then

(1) H contains a c-removable vertex of G;

(2) if H contains a vertex nonadjacent to r, then H contains a c-removable vertex nonadjacent to r; and

(3) if H contains a vertex nonadjacent to both r and s, then H contains a c-removable vertex nonadjacent to both r and s.

Proof. By definition, a simple link cannot be an induced path. Hence, by Claim 4, H contains a c-removable vertex of G. Thus (1) holds.

In order to prove (2), we assume that H contains a vertex, but no c-removable vertices, nonadjacent to r. We first prove the following subclaim in order to reach a contradiction.

Claim 5.1. There is a link contained in H. Moreover, if H contains a super heavy vertex, then there is a link contained in H and containing a super heavy vertex.

Proof. Let r' be an arbitrary vertex of H nonadjacent to r. By our assumption, r' is contained in a cut $\{r', s'\}$. Since L is simple, $s' \neq r, s$, and by Lemma 5, s' is not outside L. Now $r', s' \in H$. Let H' be the component of $G - \{r', s'\}$ contained in H. Then the subgraph induced by $H' \cup \{r', s'\}$ is a link contained in H (and containing r'). Moreover, if H contains a super heavy vertex r'', then by Claim 3, r'' is contained in a cut $\{r'', s''\}$. Similarly as in the above analysis, we get that there is a link contained in H and containing r''. □

By Lemma 8, there is a link L' contained in H such that its co-link L'_c is simple. Moreover, if H contains a super heavy vertex, L' can be chosen in such a way that it contains a super heavy vertex. Let $\{r', s'\}$ be the bolt and H' be the inside of L'. Note that every vertex in H' is nonadjacent to r. If H' contains a c-removable vertex of G, then the assertion is true. Thus, using Claim 4, we assume that L' is an induced path from r' to s'. Then similarly as in the proof of Claim 4, we get that G contains a P_5 and a Z_2 that are not o_1-heavy, a contradiction.

The third assertion can be proved similarly. We omit the details. □

Let r be a super heavy vertex of G. By Claim 3, $G-r$ is *separable*, i.e., has a cut vertex, so we can consider the *blocks* of $G-r$, i.e., the maximal subgraphs of $G-r$ without a cut vertex (these blocks are either 2-connected or isomorphic to K_2). An *end block* of $G-r$ is a block containing precisely one cut vertex of $G-r$. Note that every end block of $G-r$ contains an inner vertex (a vertex that is not the cut vertex of $G-r$ of that end block) adjacent to r. Using Lemma 2 and Lemma 3, we deduce that there are exactly two end blocks of $G-r$. This implies that the blocks of $G-r$ can be denoted as $B_0, B_1, \ldots, B_k$ with cut vertices s_i, $1 \le i \le k$, common to B_{i-1} and B_i.

Our next claim shows that $G-r$ consists of two or three blocks.

Claim 6. $k = 1$ or 2.

Proof. Suppose that $k \ge 3$. We prove the following subclaims in order to reach a contradiction. The first subclaim shows that all the super heavy vertices $\ne r$ are concentrated in one block.

Claim 6.1. All the super heavy vertices of G other than r are contained in a common end block of $G-r$.

Proof. Since r is super heavy, every other super heavy vertex is either adjacent to r or forms a super heavy pair together with r.

Using Lemma 2 and Lemma 3, note that every neighbor of r is either in B_0 or in B_k, and every vertex in $\bigcup_{i=1}^{k-1} B_i - \{s_1, s_k\}$ has at most two neighbors in common with r. This implies that every super heavy vertex other than r is either in B_0 or in B_k.

Note that $k \ge 3$. A vertex in B_0 and a vertex in B_k have at most two common neighbors, so they cannot be super heavy at the same time. Thus all the super heavy vertices of G other than r are contained in a common end block of $G-r$. □

Using Claim 6.1, without loss of generality, we assume that every super heavy vertex of G other than r is in B_0. We reach a contradiction by proving two subclaims, showing that G has an induced P_5 and an induced Z_2 that are both not o_1-heavy, respectively.

Claim 6.2. There is an induced P_5 in G that is not o_1-heavy.

Proof. Note that for every vertex s' in $B_1 - s_1$, $\{r, s'\}$ cannot be a super heavy pair, and for every vertex r' in B_k, $\{s_1, r'\}$ cannot be a super heavy pair. So either $\{r, r'\}$ is not a super heavy pair for all $r' \in B_k$ or $\{s_1, s'\}$ is not a super heavy pair for all $s' \in B_1 - s_1$. We distinguish two cases.

Case A. $\{s_1, s'\}$ is not a super heavy pair for all $s' \in B_1 - s_1$.

In this case, let x be a neighbor of s_1 in $B_0 - s_1$, let P be a shortest path of B_1 from s_1 to s_2, let Q be a shortest path of B_2 from s_2 to s_3, and let y be a neighbor of s_3 in $B_3 - s_3$. Then $xs_1Ps_2Qs_3y$ is an induced P_ℓ with $\ell \geq 5$ that is not o_1-heavy.

Case B. There is a vertex $s' \in B_1 - s_1$ such that $\{s_1, s'\}$ is a super heavy pair.

In this case, $B_1 - \{s_1, s_2\} \neq \emptyset$ and $\{r, r'\}$ is not a super heavy pair for all $r' \in B_k$. Let x be a neighbor of r in $B_0 - s_1$, let P be a shortest path of $B_k \cup \{r\}$ from r to s_k, let Q be a shortest path of B_{k-1} from s_k to s_{k-1}, and let y be a neighbor of s_{k-1} in B_{k-2} such that $y \neq s_1$. Then $xrPs_kQs_{k-1}y$ is an induced P_ℓ with $\ell \geq 5$ that is not o_1-heavy. □

Claim 6.3. There is an induced Z_2 in G that is not o_1-heavy.

Proof. Recalling that $n \geq 8$, we have $d(r) \geq 5$. This implies that r has at least two neighbors in $B_0 - s_1$ or in $B_k - s_k$. We again distinguish two cases.

Case A. r has at least two neighbors in $B_k - s_k$.

If s_k has only one neighbor x in $B_k - s_k$, then by Lemma 6, $\{r, x\}$ is a cut, a contradiction. Thus s_k has at least two neighbors in $B_k - s_k$. Let x, x' be two neighbors of s_k in $B_k - s_k$. Recall that B_k contains no super heavy vertices. By Lemma 3, $xx' \in E(G)$. Let P be a shortest path of B_{k-1} from s_k to s_{k-1}, and let y be a neighbor of s_{k-1} in B_{k-2}. Then the subgraph induced by $\{x, x'\} \cup V(P) \cup \{y\}$ is a Z_ℓ with $\ell \geq 2$ that is not o_1-heavy.

Case B. r has only one neighbor in $B_k - s_k$.

We claim that r is contained in a triangle such that the two other vertices of the triangle are in $B_0 - s_0$. Note that r has at least two neighbors in $B_0 - s_1$. Let x, x' be two neighbors of r in $B_0 - s_1$. If $xx' \in E(G)$, then $rxx'r$ is the required triangle. So we assume that $xx' \notin E(G)$. By Lemma 3, $\{x, x'\}$ is a super heavy pair. Without loss of generality, we assume that x is super heavy. Thus $d(r) + d(x) \geq n + 1$. Note that s_2 is nonadjacent to both r and x. So r and x have at least two common neighbors. Let x'' be a common neighbor of r, x other than s_1. Then $rxx''r$ is the required triangle.

Now let $rxx'r$ be a triangle such that $x, x' \in B_0 - s_1$. Let P be a shortest path of $B_k \cup \{r\}$ from r to s_k, and let y be a neighbor of s_k in B_{k-1}. Note that r has only one neighbor in B_k. No vertex in P can form a super heavy pair together with r. Thus the subgraph induced by $\{x, x'\} \cup V(P) \cup \{y\}$ is a Z_ℓ with $\ell \geq 2$ that is not o_1-heavy. □

By Claims 6.2 and 6.3, G is neither $\{K_{1,3}, P_5\}$-o_1-heavy nor $\{K_{1,3}, Z_2\}$-o_1-heavy, a contradiction. This completes the proof of Claim 6. □

By Claim 6, $G - r$ has either two or three blocks. Recalling that $d(r) \geq 5$, without loss of generality, we may assume that r has at least two neighbors in $B_0 - s_1$. Note that $\{r, x\}$ is not a cut for all $x \in B_0 - s_1$. Thus $L(r, s_1) = B_0 \cup \{r\}$ is a simple link. We distinguish two cases: $k = 2$ and $k = 1$.

Case 1. $k = 2$.

In this case, $G - r$ has three blocks B_0, B_1 and B_2. We distinguish three subcases, depending on the order of B_1 and the number of neighbors of r in B_2.

Case 1.1. $B_1 - \{s_1, s_2\} \neq \emptyset$.

We first claim that $L'(s_1, s_2) = B_1$ is a simple link. If $\{s_1, x\}$ is a cut for some $x \in B_1 - \{s_1, s_2\}$, then by Lemma 6, $\{r, x\}$ is a cut, a contradiction. Thus we assume that $\{s_1, x\}$ is not a cut for all $x \in B_1 - \{s_1, s_2\}$, and similarly, $\{s_2, x\}$ is not a cut for all $x \in B_1 - \{s_1, s_2\}$. If s_1 has only one neighbor x in $B_1 - \{s_1, s_2\}$, then by Lemma 6, s_2 has only one neighbor x in $B_1 - \{s_1, s_2\}$. This implies that $B_1 - \{s_1, s_2\}$ consists of only one vertex x; otherwise x is a cut vertex of G. If $s_1 s_2 \notin E(G)$, then by Lemma 6, $\{r, x\}$ is a cut, a contradiction. Thus we assume that $s_1 s_2 \in E(G)$. By Lemma 7, x is a c-removable vertex, and by Lemma 10, x is a d-removable vertex, a contradiction to Claim 2. Thus as we claimed, $L'(s_1, s_2) = B_1$ is a simple link.

Secondly, we claim that r has at least two neighbors in $B_2 - s_2$. Suppose to the contrary that r has only one neighbor r' in $B_2 - s_2$. Suppose first that $rs_2 \in E(G)$. If s_2 has only one neighbor r' in $B_2 - s_2$, then $B_2 - s_2$ consists of only one vertex r', and r' is a c-removable and d-removable vertex, a contradiction. If $\{s_2, r'\}$ is a cut, then let H be the component of $G - \{s_2, r'\}$ contained in $B_2 - s_2$, let x be a neighbor of s_2 in $B_1 - \{s_1, s_2\}$, and let y be a neighbor of s_2 in H. Then the subgraph induced by $\{s_2, r, x, y\}$ is a claw that is not o_1-heavy, a contradiction. Thus we assume that $rs_2 \notin E(G)$. Note that $\{r, x\}$ is not a super heavy pair for every $x \in B_2$. Let x be a neighbor of r in $B_0 - s_1$, let P be a shortest path of B_2 from r' to s_2, and let y be a neighbor of s_2 in $B_1 - \{s_1, s_2\}$. Then $xrr'Ps_2y$ is an induced P_ℓ for $\ell \geq 5$ that is not o_1-heavy. At the same time, similarly as in Case B of Claim 6.3, we can prove that r is contained in a triangle $rxx'r$ with $x, x' \in B_0 - s_1$. Let y be a neighbor of r' in $B_2 - r'$. Then the subgraph induced by $\{x, x', r, r', y\}$ is a Z_2 that is not o_1-heavy. Thus G is neither $\{K_{1,3}, P_5\}$-o_1-heavy nor $\{K_{1,3}, Z_2\}$-o_1-heavy, a contradiction. So as we claimed, r has at least two neighbors in $B_2 - s_2$. Note that $\{r, x\}$ is not a cut for all $x \in B_2 - s_2$. Hence $L''(s_2, r) = B_2 \cup \{r\}$ is a simple link.

We conclude that G consists of three simple links $L = L(r, s_1)$, $L' = L'(s_1, s_2)$ and $L'' = L''(s_2, r)$.

Suppose that there is a vertex inside L nonadjacent to r. Using Claim 5, let x be a c-removable vertex inside L nonadjacent to r, and let y be a c-removable vertex in L''. Then by Lemma 9, $\{x, y\}$ is a c-removable pair, and by Lemma 10, $\{x, y\}$ is a d-removable pair, a contradiction. Thus we deduce that r is adjacent to every vertex inside L. Similarly, we can prove that s_1 is adjacent to every vertex inside L', and s_2 is adjacent to every vertex inside L''.

We claim that L contains a path from r to s_1 of length k for all $k \in [2, |V(L)| - 1]$. Recall that G is hamiltonian and that $\{r, s_1\}$ is a cut of G. There is a Hamilton path of L from r to s_1. Let $P = rx_1x_2 \cdots x_js_1$ be a Hamilton path of L, where $j = |V(L)| - 2$. Then $rx_{j-k+2} \cdots x_js_1$ is a path of L from r to s_1 of length k.

Thus as we claimed, L contains a path from r to s_1 of length k for all $k \in [2, |V(L)| - 1]$. Similarly, L' contains a path from s_1 to s_2 of length k for all $k \in [2, |V(L')| - 1]$, and L'' contains a path from s_2 to r of length k for all $k \in [2, |V(L'')| - 1]$. Thus G contains a cycle of length k for all $k \in [6, n]$.

Case 1.2. $B_1 - \{s_1, s_2\} = \emptyset$ and r has at least two neighbors in $B_2 - s_2$.

Note that $\{r, x\}$ is not a cut for all $x \in B_2 - s_2$. Hence $L'(r, s_2) = B_2 \cup \{r\}$ is a simple link. So G consists of two simple links $L = L(r, s_1)$ and $L' = L'(r, s_2)$, and an edge s_1s_2.

Similarly as in the proof of Case 1.1, we get that r is adjacent to every vertex inside L, and L contains a path from r to s_1 of length k for all $k \in [2, |V(L)|-1]$. Similarly, r is adjacent to every vertex inside L', and L' contains a path from r to s_2 of length k for all $k \in [2, |V(L')| - 1]$. Thus G contains a cycle of length k for all $k \in [5, n]$.

Case 1.3. $B_1 - \{s_1, s_2\} = \emptyset$ and r has only one neighbor in $B_2 - s_2$.

Let r' be the neighbor of r in $B_2 - s_2$. Suppose first that $B_2 - s_2$ consists of only one vertex r'. If $rs_2 \in E(G)$, then r' is a c-removable vertex and a d-removable vertex, a contradiction. If $rs_2 \notin E(G)$, then $\{r', s_2\}$ is a c-removable pair and a d-removable pair, also a contradiction. Thus we assume that $B_2 - s_2$ has at least two vertices. By Lemma 6, $\{r', s_2\}$ is a cut and $L'(r', s_2) = B_2$ is a link. Now we get that G consists of two links $L = L(r, s_1)$ and $L' = L'(r', s_2)$, and two edges rr' and s_1s_2 (and maybe an additional edge rs_2).

We claim that L' is 2-connected. If $r's_2 \in E(G)$, then by Lemma 7, L' is 2-connected. Thus we assume that $r's \notin E(G)$. If s_2 has only one neighbor x inside L' or $\{s_2, x\}$ is a cut for some x inside L', then by Lemma 6, $\{r, x\}$ is a cut, a contradiction. Thus L' is simple, and by Lemma 7, L' is 2-connected.

Note that L' is not a path. Using Claim 4, let x be a c-removable vertex inside L, and let y be a c-removable vertex inside L'. Then by Lemma 9, $\{x, y\}$ is a c-removable pair, and by Lemma 10, $\{x, y\}$ is a d-removable pair, a contradiction.

This completes the proof of Case 1.

Case 2. $k = 1$.

In this case, $G - r$ has only two blocks B_0 and B_1. We again distinguish three subcases according to the order and the number of neighbors of r in B_1.

Case 2.1. r has at least two neighbors in $B_1 - s_1$.

Note that $\{r, x\}$ is not a cut for all $x \in B_1 - s_1$. We have that $L'(r, s_1) = B_1 \cup \{r\}$ is a simple link. So G consists of two simple links $L = L(r, s_1)$ and $L' = L'(r, s_1)$.

If there is a vertex inside L nonadjacent to both r and s_1, then using Claim 5, let x be a c-removable vertex inside L nonadjacent to both r and s_1, and let

y be a c-removable vertex inside L'. Then by Lemma 9, $\{x, y\}$ is a c-removable pair, and by Lemma 10, $\{x, y\}$ is a d-removable pair, a contradiction. Thus we assume that every vertex inside L is either adjacent to r or to s_1, and similarly, every vertex inside L' is either adjacent to r or to s_1.

If there is a super heavy vertex r' inside L, then by Claim 3, r' is contained in a cut $\{r', s'\}$. Since L is simple, $s' \neq r, s$, and by Lemma 5, s' is inside L. Using Lemma 4, let H be the component of $G - \{r', s'\}$ contained in $B_0 - s_1$. Then every vertex in H is nonadjacent to both r and s_1, a contradiction. Thus we assume that there are no super heavy vertices inside L.

Note that there are at least two vertices inside L. We can divide the inside of L into two nonempty subsets H and H' such that every vertex of H is adjacent to r and every vertex of H' is adjacent to s_1. Let xy be an edge connecting H and H', where $x \in H$ and $y \in H'$. Note that there are no super heavy vertices in H. By Lemma 3, H is a clique. Thus $H \cup \{r\}$ contains a path from r to x of length k for all $k \in [1, |V(H)|]$, and similarly, $H' \cup \{s_1\}$ contains a path from y to s_1 of length k for all $k \in [1, |V(H')|]$. Hence L contains a path from r to s_1 of length k for all $k \in [3, |V(L)| - 1]$, and similarly, L' contains a path from r to s_1 of length k for all $k \in [3, |V(L')| - 1]$. So G contains a cycle of length k for all $k \in [6, n]$.

Case 2.2. $B_1 - s_1$ has at least two vertices and r has only one neighbor in $B_1 - s_1$.

Let r' be the neighbor of r in $B_1 - s_1$. By Lemma 6, $\{r', s_1\}$ is a cut and $L' = L'(r', s_1) = B_1$ is a link. If L' is simple, then G consists of two simple links $L = L(r, s_1)$ and $L'(r', s_1)$, and an edge rr'. Then as in Case 1.2, we can prove that G contains a cycle of length k for all $k \in [5, n]$. Thus we assume that L' is not simple.

If $\{s_1, x\}$ is a cut for some x inside L', then by Lemma 6, $\{r, x\}$ is a cut, a contradiction. Thus we assume that $\{s_1, x\}$ is not a cut for all x inside L'. Note that L' is not simple. Now s_1 has only one neighbor s' inside L. If $r's_1 \notin E(G)$, then $\{r, s'\}$ is a cut, a contradiction. Thus we assume that $r's_1 \in E(G)$. If there is only one vertex r' inside L', then by Lemma 7, r' is a c-removable vertex, and by Lemma 10, r' is a d-removable vertex, a contradiction. Thus we assume that there are at least two vertices inside L'. By Lemma 6, $\{r', s'\}$ is a cut and $L''(r', s') = B_1 - s_1$ is a link. Thus G consists of two links $L(r, s_1)$ and $L'' = L''(r', s')$, and three edges rr', s_1s' and $r's_1$. Similarly as in Case 1.3, we can prove that L'' is 2-connected. Using Claim 4, let x be a c-removable vertex inside L, and let y be a c-removable vertex inside L''. Then by Lemma 9, $\{x, y\}$ is a c-removable pair, and by Lemma 10, $\{x, y\}$ is a d-removable pair, a contradiction.

Case 2.3. $B_1 - s_1$ has only one vertex.

Let y be the vertex of $B_1 - s_1$. Recall that L is simple. Now y is a c-removable vertex. Using Claim 3, we get that y is not a d-removable vertex. By Lemma 10, $\{r, s_1\}$ is a super heavy pair.

First we assume that there is a vertex inside L nonadjacent to both r and s_1. By Claim 5, let x be a c-removable vertex inside L nonadjacent to both r and s_1. Then by Lemma 10, $\{x, y\}$ is a d-removable pair.

We claim that $\{x, y\}$ is a c-removable pair. Let z be an arbitrary vertex of $G - \{x, y\}$. We prove that $G' = G - \{x, y, z\}$ is connected. If $z = r$ or s_1, then without loss of generality, we assume that $z = r$. Then for every vertex v inside L with $v \neq x$, since x is c-removable and $\{r, x\}$ is not a cut, there is a path P of $G - \{r, x\}$ from v to s_1. Clearly, P does not pass through y. This implies that v and s_1 are connected by the path P in G'. Thus G' is connected. So we assume that $z \neq r, s_1$ and then z is inside L. Then for every vertex v inside L with $v \neq x, z$, since x is c-removable and $\{x, z\}$ is not a cut, there is a path P of $G - \{x, z\}$ from v to r or s with all internal vertices inside L. This implies that v and r or s are connected by the path P in G'. Recall that $\{r, s_1\}$ is a super heavy pair. By Lemma 3, r and s_1 have a common neighbor t other than y and z. Thus r and s_1 are connected by the path rts_1 in G'. This implies that G' is connected. Thus as we claimed, $\{x, y\}$ is a c-removable pair, and recalling that $\{x, y\}$ is a d-removable pair too, we obtain a contradiction.

In the remaining case, we assume that every vertex inside L is either adjacent to r or to s_1. Similarly as in Case 2.1, we can prove that there are no super heavy vertices inside L, and that L contains a path from r to s_1 of length k for all $k \in [3, |V(L)| - 1]$. Thus G contains a cycle of length k for all $k \in [5, n]$.

This completes the proof of Theorem 8.5.

Chapter 9

Heavy pairs for path partition optimality

9.1 Introduction

A *path partition* of a graph G is the union of some pairwise vertex-disjoint paths such that every vertex of G is contained in one of the paths. If G is a nonhamiltonian graph, then the *path partition number* of G, denoted by $\pi(G)$, is the minimum number of paths in a path partition of G; if G is hamiltonian, then we define $\pi(G) = 0$. Alternatively, $\pi(G)$ is the minimum number of edges we have to add to G to turn it into a hamiltonian graph, except for degenerate cases. Note that $\pi(K_1) = \pi(K_2) = 1$ and $\pi(2K_1) = 2$.

The *separable degree* of a graph G, denoted by $\sigma(G)$, is defined as the minimum number of edges one has to add to G to turn it into a 2-connected graph, again except for degenerate cases. We define $\sigma(K_1) = \sigma(K_2) = 1$ and $\sigma(2K_1) = 2$. Note that any 2-connected graph has separable degree 0 and any disconnected graph has separable degree at least 2.

It is not difficult to see that for every graph G, $\pi(G) \geq \sigma(G)$: if G has only one or two vertices, then the result is trivially true. If G has at least three vertices, then the result can be obtained by the fact that a hamiltonian graph is necessarily 2-connected.

We call a graph *path partition optimal* if its path partition number is equal to its separable degree. It is clear from the above definitions that K_1, K_2 and $2K_1$ are path partition optimal.

In this final chapter of the thesis, we consider subgraph conditions for path partition optimality of graphs.

We first prove an extension of Ore's Theorem for hamiltonicity.

For a graph G that is not complete, we define $\sigma_2(G)$ as the minimum degree sum of any two nonadjacent vertices of G. If G is complete, then we define $\sigma_2(G) = \infty$. We repeat Ore's Theorem for convenience. Here $n(G)$ denotes the number of vertices of G.

Theorem 9.1 (Ore [30]). *Let G be a 2-connected graph. If $\sigma_2(G) \geq n(G)$, then G is hamiltonian.*

As an extension of Theorem 9.1, we obtain the following result.

Theorem 9.2. *Let G be a graph. If $\sigma_2(G) \geq n(G) - \sigma(G)$, then G is path partition optimal.*

Proof. If $n(G) \leq 2$, then the result is trivially true. So we assume that $n(G) \geq 3$. If G is 2-connected, then the result follows from Theorem 9.1. So we assume that $\sigma(G) \geq 1$.

Let Y be a set of vertices such that $Y \cap V(G) = \emptyset$ and $|Y| = \sigma(G)$. We construct a graph G' such that $V(G') = V(G) \cup Y$ and $E(G') = E(G) \cup \{uv : u \in Y, v \in Y \cup V(G)\}$. Note that Y is a clique in G' and that every vertex of Y is adjacent to every vertex of $V(G)$.

If $\sigma(G) \geq 2$, then $|Y| \geq 2$ and G' is 2-connected; if $\sigma(G) = 1$, then G is connected and G' is 2-connected. Thus in any case, G' is 2-connected.

Let u and v be two nonadjacent vertices in G'. Then $u, v \in V(G)$ and $d(u) + d(v) \geq n(G) - \sigma(G)$. Noting that $d_{G'}(u) = d_G(u) + \sigma(G)$ and $d_{G'}(v) = d_G(v) + \sigma(G)$, we get $d_{G'}(u) + d_{G'}(v) \geq n(G) + \sigma(G) = n(G')$. By Theorem 9.1, G' is hamiltonian.

Let C be a Hamilton cycle of G'. Then $C - Y$ is a path partition of G with at most $\sigma(G)$ paths. This implies that $\pi(G) \leq \sigma(G)$, and hence that $\pi(G) = \sigma(G)$. Thus G is path partition optimal. □

Note that a graph G is hamiltonian if and only if $\pi(G) = 0$; and that G is traceable but nonhamiltonian if and only if $\pi(G) = 1$. If a graph G is connected and P_3-free, then it is a complete graph and it is trivially traceable (and hamiltonian if $n(G) \geq 3$). In fact, as we have seen before P_3 is the only single subgraph with this property.

The following theorem on forbidden pairs of subgraphs for hamiltonicity is well-known, and is repeated here for convenience.

Theorem 9.3 (Duffus, Jacobson and Gould [21]). *Let G be a $\{K_{1,3}, N\}$-free graph. Then*
(1) *if G is connected, then G is traceable;*
(2) *if G is 2-connected then G is hamiltonian.*

As noted before, if H is an induced subgraph of N, then $\{K_{1,3}, H\}$-free yields the same conclusions in the above theorem, and Faudree et al. proved that these are the only forbidden pairs with this property.

Theorem 9.4 (Faudree and Gould [24]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$, P_4, Z_1, B or N.*

If a disconnected graph G is P_3-free, then every component of G is a clique. Clearly such a graph has a separable degree equal to its number of components and equal to its path partition number, and hence is path partition optimal. In fact, we will prove the following.

Theorem 9.5. *The only connected graph S such that a graph G being S-free implies G is path partition optimal is P_3.*

Proof. We first prove the 'if' part of the theorem. If G has only one or two vertices, then the result is trivially true. If G is a connected P_3-free graph with at least three vertices, then $\pi(G) = 0$ and the result is also true. Next we assume that G is a disconnected P_3-free graph with at least three vertices. Then every component of G is a clique. Clearly such a graph has a separable degree equal to its number of components and equal to its path partition number, and hence is path partition optimal.

Now we prove the 'only-if' part of the theorem. Let G_1 be a disconnected graph with k components each of which is a $K_{1,3}$; and let G_2 be a disconnected graph with k components each of which is an N. Note that $\pi(G_1) = \pi(G_2) = 2k$ and $\sigma(G_1) = \sigma(G_2) = \lceil 3k/2 \rceil$. Neither G_1 nor G_2 is path partition optimal.

Thus S is a common connected induced subgraph of G_1 and G_2. Note that the only common connected induced subgraph of G_1 and G_2 (on at least three vertices) is P_3, Thus we conclude that $S = P_3$. □

It is not difficult to see that a $\{K_{1,3}, N\}$-free graph with connectivity 1 has separable degree 1. Thus Theorem 9.3 implies that any connected $\{K_{1,3}, N\}$-free graph is path partition optimal. In fact, this statement is also true for disconnected graphs.

Theorem 9.6. *Let G be a $\{K_{1,3}, N\}$-free graph. Then G is path partition optimal.*

Proof. If G is connected, then by the above analysis the result is true. Suppose now that G has $k \geq 2$ components.

Let H be an arbitrary component of G. Then H is connected and $\{K_{1,3}, N\}$-free. By Theorem 9.3, H contains a Hamilton path. Note that the set of Hamilton paths of the components of G is a path partition of G. Hence $\pi(G) = k$.

Let G' be a 2-connected spanning supergraph of G. Then every component of G is joined to other components by at least two additional edges in G'. This implies that G' has at least k additional edges. Hence $\sigma(G) \geq k$.

Thus $\pi(G) \leq \sigma(G)$ and we conclude that $\pi(G) = \sigma(G)$, so G is path partition optimal. □

For pairs of forbidden subgraphs, we obtain the following counterpart of Theorem 9.4.

Theorem 9.7. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a graph. Then G being $\{R, S\}$-free implies G is path partition optimal if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$, P_4, Z_1, B or N.*

Proof. The 'if' part of the theorem can be deduced from Theorem 9.6 immediately. Now we prove the 'only-if' part of the theorem. Let G_1 and G_2 be two graphs as in the proof of Theorem 9.5. Since neither G_1 nor G_2 is path partition optimal, G_1 and G_2 must contain R or S as an induced subgraph. Note that the maximal connected induced subgraphs of G_1 and G_2 are $K_{1,3}$ and N, respectively. This implies that (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of N. □

Before stating and proving the counterparts of the above theorems for heavy subgraphs, we introduce some additional terminology and notation.

Let G be a graph and let G' be an induced subgraph of G. We define the *heft* of G' in G, denoted by $h_G(G')$ (or briefly, $h(G')$), as the maximum degree sum of two nonadjacent vertices in $V(G')$. If G' is a clique, then we define $h(G') = 0$. For a given graph H, the *H-heft index* of G, denoted by $\eta_H(G)$, is the minimum heft of an induced subgraph of G isomorphic to H. If G is H-free, then we define $\eta_H(G) = \infty$. Note that if H_1 is an induced subgraph of H_2, then $\eta_{H_1}(G) \leq \eta_{H_2}(G)$. By the above definitions, a graph G with $\eta_H(G) \geq n(G) + k$ is an H-o_k-heavy graph.

With respect to heavy subgraph conditions for path partition optimality, we obtained the following extensions of Theorems 9.5 and 9.7 involving the heft index.

Theorem 9.8. *The only connected graph S such that a graph G satisfying $\eta_S(G) \geq n(G) - \sigma(G)$ implies G is path partition optimal is P_3.*

Theorem 9.9. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a graph. Then $\eta_R(G) \geq n(G) - \sigma(G)$ and $\eta_S(G) \geq n(G) - \sigma(G)$ implies G is path partition optimal if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$, P_4, Z_1, B or N.*

Note that if a graph G is H-free, then $\eta_H(G) = \infty \geq n(G) - \sigma(G)$. Thus the 'only if' part of Theorems 9.8 and 9.9 can be deduced from Theorems 9.5 and 9.7, respectively. For the 'if' part of Theorem 9.9, it is sufficient to prove the following.

Theorem 9.10. *Let G be a graph. If $\eta_{K_{1,3}}(G) \geq n(G) - \sigma(G)$ and $\eta_N(G) \geq n(G) - \sigma(G)$, then G is path partition optimal.*

Note that P_3 is an induced subgraph of $K_{1,3}$ and N. Thus $\eta_{P_3}(G) \leq \eta_{K_{1,3}}(G)$ and $\eta_{P_3}(G) \leq \eta_N(G)$, so the 'if' part of Theorem 9.8 can also be deduced from Theorem 9.10. Before we are going to prove Theorem 9.10 in the final section of this chapter, we introduce some additional terminology and prove some useful auxiliary results in the next section.

9.2 Some preliminaries

Let G be a graph. For a subgraph B of G, when no confusion can arise, we also use B to denote its vertex set; similarly, for a subset C of $V(G)$, we also use C to denote the subgraph induced by C.

We use $\omega(G)$ to denote the number of components of G. A vertex v of G is called a *cut vertex* of G if $\omega(G-v) > \omega(G)$. A graph is said to be *separable* if it is disconnected or has at least one cut vertex. A maximal non-separable subgraph of G is called a *block* of G. Thus a block is either a non-separable component of G or contains at least one cut vertex of G. A block which contains exactly one cut vertex of G is called an *end block* of G; and a block which is neither a non-separable component nor an end block is called an *inner block* of G. A vertex in an end block which is not a cut vertex is called an *inner vertex* of the end block. So an end block has at least one inner vertex and an inner block contains at least two cut vertices of G. A component with only one vertex is called a *trivial* component (the vertex of such a component is called an *isolated vertex*), and a block isomorphic to K_2 is called a *trivial* block (the inner vertex of such an end block is called a *leaf*). Note that every edge of G is contained in exactly one block.

Note that a vertex is a cut vertex if and only if it is contained in at least two blocks. We call a vertex a *chop vertex* if it is contained in at least three blocks. Let x be a chop vertex, and let y, y', y'' be three neighbors of x in distinct components of $G - x$. Then the subgraph induced by $\{x, y, y', y''\}$ is a claw. A claw of such a type is called a *chop claw.*

Lemma 1. *Let G be a graph, let x be a cut vertex of G, and let H be a component of $G - x$ containing a neighbor of x. Then $G' = H \cup \{x\}$ contains exactly one block containing x and at least one end block of G.*

Proof. Note that $G' - x = H$ is connected, so x is not a cut vertex of G'. The unique block of G' containing x is also a block of G.

If G' is non-separable, then it is an end block of G. If G' is separable, then let H' be an end block of G' not containing x (note that a separable graph contains at least two end blocks). Then H is an end block of G. □

We use $\mathrm{sco}(G)$ to denote the number of separable components of G, $\mathrm{nco}(G)$

to denote the number of non-separable components of G, $\mathrm{ebl}(G)$ to denote the number of end blocks of G, and $\mathrm{cve}(G)$ to denote the number of chop vertices of G. We define

$$\sigma'(G) = \begin{cases} \mathrm{ebl}(G) + \mathrm{nco}(G) - \mathrm{sco}(G), & \text{if } \mathrm{cve}(G) \leq 1; \\ \mathrm{ebl}(G) + \mathrm{nco}(G) - \mathrm{sco}(G) - 1, & \text{otherwise,} \end{cases}$$

and

$$\sigma''(G) = \lceil \mathrm{ebl}(G)/2 \rceil + \mathrm{nco}(G).$$

Lemma 2. *For every graph G, $\sigma''(G) \leq \sigma(G) \leq \sigma'(G)$.*

Proof. If G has only one or two vertices, or G is 2-connected, then the result is trivially true. Next we assume that $n(G) \geq 3$ and that G is separable.

Let G' be a 2-connected spanning supergraph of G and let $E' = E(G') \setminus E(G)$. Note that for every end block of G, there is at least one edge of E' incident with some inner vertex of the end block, and for every non-separable component of G, there are at least two edges of E' incident with some vertices of the component. Hence $|E'| \geq \lceil \mathrm{ebl}(G)/2 \rceil + \mathrm{nco}(G)$, implying that $\sigma''(G) \leq \sigma(G)$.

We now prove $\sigma(G) \leq \sigma'(G)$ by induction on $n(G)(n(G)-1)/2 - |E(G)|$. Recall that the result is true if G is 2-connected. So we assume that G is a separable graph.

First we assume that G is disconnected. Let H and H' be two components of G.

If both H and H' are separable, then let x be an inner vertex of some end block in H, and let x' be an inner vertex of some end block in H'. Let G' be the graph obtained from G by adding the edge xx'. By the induction hypothesis, $\sigma(G') \leq \sigma'(G')$. Note that $\mathrm{ebl}(G') = \mathrm{ebl}(G) - 2$, $\mathrm{nco}(G') = \mathrm{nco}(G)$, $\mathrm{sco}(G') = \mathrm{sco}(G) - 1$ and $\mathrm{cve}(G') = \mathrm{cve}(G)$. Thus $\sigma'(G') = \sigma'(G) - 1$. It is not difficult to see that $\sigma(G) \leq \sigma(G') + 1$. So we get that $\sigma(G) \leq \sigma'(G)$.

If H is separable and H' is non-separable, then let x be an inner vertex of some end block in H, and let x' be an arbitrary vertex of H'. Let G' be the graph obtained from G by adding the edge xx'. By the induction hypothesis, $\sigma(G') \leq \sigma'(G')$. Note that $\mathrm{ebl}(G') = \mathrm{ebl}(G)$, $\mathrm{nco}(G') = \mathrm{nco}(G) - 1$, $\mathrm{sco}(G') = \mathrm{sco}(G)$ and $\mathrm{cve}(G') = \mathrm{cve}(G)$. Thus $\sigma'(G') = \sigma'(G) - 1$. It is not difficult to see that $\sigma(G) \leq \sigma(G') + 1$. So we get that $\sigma(G) \leq \sigma'(G)$.

The case that H is non-separable and H' is separable follows by symmetry.

If both H and H' are non-separable, then let x be an arbitrary vertex of H, and let x' be an arbitrary vertex of H'. Let G' be the graph obtained from G by adding the edge xx'. By the induction hypothesis, $\sigma(G') \le \sigma'(G')$. If both H and H' are trivial, then $\mathrm{ebl}(G') = \mathrm{ebl}(G)$, $\mathrm{nco}(G') = \mathrm{nco}(G) - 1$, $\mathrm{sco}(G') = \mathrm{sco}(G)$ and $\mathrm{cve}(G') = \mathrm{cve}(G)$; otherwise, $\mathrm{ebl}(G') = \mathrm{ebl}(G) + 2$, $\mathrm{nco}(G') = \mathrm{nco}(G) - 2$, $\mathrm{sco}(G') = \mathrm{sco}(G) + 1$ and $\mathrm{cve}(G') = \mathrm{cve}(G)$. In both cases, $\sigma'(G') = \sigma'(G) - 1$. It is not difficult to see that $\sigma(G) \le \sigma(G') + 1$. So we get that $\sigma(G) \le \sigma'(G)$.

Next we assume that G is connected. Thus $\mathrm{nco}(G) = 0$ and $\mathrm{sco}(G) = 1$. If G contains at least two chop vertices, then let x and x' be two chop vertices of G with the distance between them as short as possible. Let H be a component of $G - x$ not containing x', and let H' be a component of $G - x'$ not containing x. By Lemma 1, $H \cup \{x\}$ ($H' \cup \{x'\}$) contains at least one end block of G. Let y be a vertex of H that is an inner vertex of an end block, and let y' be a vertex of H' that is an inner vertex of an end block. Let G' be the graph obtained from G by adding the edge yy'. By the induction hypothesis, $\sigma(G') \le \sigma'(G')$. Since x and x' are cut vertices of G', $\mathrm{nco}(G') = 0$ and $\mathrm{sco}(G') = 1$. Note that y and y' are contained in a common block of G', and this block is an inner block (since it contains at least two cut vertices x and x'). Now $\mathrm{ebl}(G') = \mathrm{ebl}(G) - 2$. If G' also contains at least two chop vertices, then $\sigma'(G') = \sigma'(G) - 2$; otherwise $\sigma'(G') = \sigma'(G) - 1$. Thus $\sigma'(G') \le \sigma'(G) - 1$. It is not difficult to see that $\sigma(G) \le \sigma(G') + 1$. So we get that $\sigma(G) \le \sigma'(G)$.

Finally we assume that G contains at most one chop vertex. If G has a chop vertex, then let x be the chop vertex of G; if G contains no chop vertices, then let x be a cut vertex of G. Let H_i, $1 \le i \le k$, be the components of $G - x$. Clearly, $H_i \cup \{x\}$, $1 \le i \le k$, contains exactly one end block of G; otherwise, there would be a second chop vertex. Thus we have $\mathrm{ebl}(G) = k$, and $\sigma'(G) = k - 1$. Let y_i be a vertex of H_i that is an inner vertex of an end block. Let G' be the graph obtained from G by adding edges $y_i y_k$, $1 \le i \le k - 1$. Then G' is 2-connected. This implies that $\sigma(G) \le k - 1 = \sigma'(G)$. □

Let G be a graph and let v be a vertex of G. We call v a *good vertex* if v is contained in at most one end block of G.

Lemma 3. *Let G be a graph. If G has a chop vertex, then G contains a chop claw with at least two good end-vertices.*

Proof. Let H be a component of G that contains a chop vertex. If every vertex in H is good, then let x be a chop vertex in H, let C, C' and C'' be three components of $H-x$, and let y, y' and y'' be neighbors of x in C, C' and C'', respectively. Then the subgraph induced by $\{x, y, y', y''\}$ is the required claw.

Now we assume that there is a bad vertex x in H. Let B and B' be two end blocks containing x, and let $C = B - x$, $C' = B' - x$. If H consists of $B \cup B'$, then there are no chop vertices in H, a contradiction. This implies that there is at least a third component C'' of $H - x$ other than C and C'. Let y, y' and y'' be neighbors of x in C, C' and C'', respectively. Note that y and y' are good vertices. Now the subgraph induced by $\{x, y, y', y''\}$ is the required claw. □

Adopting the terminology of [26], we say that a graph is a *block-chain* if it is non-separable or it has at least one cut vertex and has exactly two end blocks.

A vertex with degree at least $(n(G) - \sigma(G))/2$ is called a *critical vertex* and a pair of nonadjacent vertices with degree sum at least $n(G) - \sigma(G)$ is called a *critical pair*. Thus a critical pair contains at least one critical vertex.

We finish this section by recalling the following two theorems, that have been proved in earlier chapters of the thesis, for later reference.

Theorem 9.11. *Let G be a 2-connected graph. If G is $\{K_{1,3}, N\}$-heavy, then G is hamiltonian.*

Theorem 9.12. *Let G be a block-chain. If G is $\{K_{1,3}, N\}$-o_{-1}-heavy, then G is traceable.*

From these two theorems, we immediately get that a block-chain G with $\eta_{K_{1,3}}(G) \geq n(G) - \sigma(G)$ and $\eta_N(G) \geq n(G) - \sigma(G)$ is path partition optimal.

9.3 Proof of Theorem 9.10

Let G be a graph on $n(G)$ vertices, and assume that $\eta_{K_{1,3}}(G) \geq n(G) - \sigma(G)$ and $\eta_N(G) \geq n(G) - \sigma(G)$. We are going to prove that G is path partition optimal by induction on $n(G)$. If G has only one or two vertices, then the result is trivially true. So we assume that $n(G) \geq 3$.

If G is $\{K_{1,3}, N\}$-free, then the result is true by Theorem 9.5. So we assume that there is at least one induced subgraph isomorphic to $K_{1,3}$ or N. Since

$\eta_{K_{1,3}}(G) \geq n(G) - \sigma(G)$ and $\eta_N(G) \geq n(G) - \sigma(G)$, there is at least one critical pair in G. We distinguish two cases: G has a chop vertex or does not have a chop vertex.

Case 1. G has a chop vertex.

By Lemma 3, there exists a chop claw, say induced by $\{x, y, y', y''\}$, such that at least two end-vertices, say y and y', are good. Let C, C' and C'' be the components of $G - x$ containing y, y' and y'', respectively, and let $D = C - y$, $D' = C' - y'$ and $D'' = C'' - y''$. Without loss of generality, we assume that $d(y) = \min\{d(v) : v \in N_C(x)\}$ and $d(y') = \min\{d(v') : v' \in N_{C'}(x)\}$. Let H be the component of G containing x.

Since $\eta_{K_{1,3}}(G) \geq n(G) - \sigma(G)$, there is a critical pair in $\{y, y', y''\}$. We again distinguish two cases: $\{y, y'\}$ is a critical pair or at least one of $\{y, y''\}$ and $\{y', y''\}$ is a critical pair.

Case 1.1. $\{y, y'\}$ is a critical pair.

By Lemma 2, $d(y) + d(y') + \sigma'(G) \geq n(G)$.

Let U be the union of blocks containing y, and let U' be the union of blocks containing y'. Note that $d(y) \leq n(U) - 1$ and $d(y') \leq n(U') - 1$. For every end block of G not containing y and y', we choose an inner vertex of it as the representative of this end block; and for every non-separable component of G, we choose a vertex of it as the representative of this non-separable component. Let S and T be the set of all these representatives of end blocks and non-separable components, respectively. Thus $\text{ebl}(G) \leq |S| + 2$ and $\text{nco}(G) = |T|$. Note that H is a separable component, and x is a chop vertex, of G, so $\text{sco}(G) \geq 1$ and $\text{cve}(G) \geq 1$. Thus $d(y) + d(y') + \sigma'(G) \leq n(C) + n(C') + |S| + 2 + |T| - 1 \leq n(U - x) + n(U' - x) + |S| + |T| + |\{x\}| \leq n$.

We conclude that $d(y) + d(y') + \sigma'(G) = n(G)$. Note that this equation holds only if y is adjacent to every vertex in D, y' is adjacent to every vertex in D', every end block not containing y and y' and every non-separable component of G is trivial, there is only one separable component H and only one chop vertex x, y and y' are both contained in an end block, and every vertex of G is either an isolated vertex, a leaf or adjacent to y or y'. Moreover, this implies that $U - x = C$ and $U' - x = C'$.

Let $s = |S|$ and $t = |T|$. Note that $\sigma'(G) = s + 2 + t - 1 = s + t + 1$. So $\sigma(G) \leq s + t + 1$. We claim that $\sigma(G) = s + t + 1$. Let G' be a 2-connected spanning supergraph of G. Then $G' - x$ is connected. Since $G - x$ has $s+t+2$ components, $e(G'-x) - e(G-x) \geq s+t+1$. This implies that $\sigma(G) \geq s+t+1$. Thus as we claimed, $\sigma(G) = s + t + 1$.

Next we claim that $C \cup \{x\}$ contains a Hamilton path starting from x. If C contains only one or two vertices, then the result is trivially true. So we assume that C has at least three vertices.

If there is a second neighbor z of x in C, then z is a good vertex; otherwise there would be some vertex in C nonadjacent to y. Note that $d(y) \leq d(z)$. Similarly as in the above analysis, we obtain that y is adjacent to all the vertices of $C - z$. Thus C is 2-connected. If there is only one neighbor y of x in C, then note that y is adjacent to all the vertices in D, and y is the only possible cut vertex of C. But this implies that y is contained in at least two end blocks, contradicting that y is a good vertex. So in any case, we have that C is 2-connected.

Now we claim that C is $\{K_{1,3}, N\}$-heavy. Note that if some vertex in C is adjacent to x, then it is also adjacent to all other vertices of C. Let u, v be two nonadjacent vertices such that $d(u) + d(v) \geq n(G) - \sigma(G)$. Then $ux, vx \notin E(G)$. Thus $d_C(u) + d_C(v) = d(u) + d(v) \geq n(G) - \sigma(G) = n(G) - (s + t + 1) = n(C)$. Since $\eta_{K_{1,3}}(G) \geq n(G) - \sigma(G)$ and $\eta_N(G) \geq n(G) - \sigma(G)$, we conclude that C is $\{K_{1,3}, N\}$-heavy.

By Theorem 9.11, C has a Hamilton cycle. This implies that $C \cup \{x\}$ has a Hamilton path starting from x, and similarly, $C' \cup \{x\}$ contains a Hamilton path starting from x.

Let Q and Q' be the Hamilton path of $C \cup \{x\}$ and $C' \cup \{x\}$, respectively, starting from x. Then QxQ' and all the isolated vertices of $S \cup T$ form a path partition of G. Hence $\pi(G) \leq s+t+1 = \sigma(G)$. So G is path partition optimal.

Case 1.2. $\{y, y''\}$ or $\{y', y''\}$ is a critical pair.

Without loss of generality, we assume that $\{y, y''\}$ is a critical pair. If y'' is a good vertex, then we can prove this case similarly as Case 1.1. So we assume that y'' is contained in at least two end blocks. Moreover, if there is a chop claw with three good end-vertices, we can also prove that G is path partition optimal as before. Thus we assume that there are no such chop claws. If some vertex is contained in at least three end blocks, then there will be a chop claw with three good end-vertices. Thus we assume that every bad vertex of G is contained in exactly two end blocks.

Let B be the block containing the edge xy''. Then B is an inner block. Let $V(B) = \{x_1, x_2, \ldots, x_k\}$, where $x_1 = x$ and $x_2 = y''$.

Note that x_2 is bad. Here we prove that if some vertex x_i of B is bad, then every neighbor of it in B, say x_j, is also bad. Assume that x_j is good. Note that x_i is contained in exactly two end blocks. Let y_i and y'_i be two neighbors of x_i in distinct end blocks. Then the subgraph induced by $\{x_i, y_i, y'_i, x_j\}$ is a chop claw with three good end-vertices, a contradiction. This implies that every vertex in B is bad.

For the bad vertex x_i, let B_i, B'_i be the two end blocks containing x_i, and let $C_i = B_i - x_i$, $C'_i = B'_i - x_i$, and y_i and y'_i be two neighbors of x_i in C_i and C'_i, respectively. Without loss of generality, we assume that $d(y_i) = \min\{d(v_i) : v_i \in N_{C_i}(x_i)\}$, $d(y'_i) = \min\{d(v'_i) : v'_i \in N_{C'_i}(x_i)\}$ and $d(y_i) \geq d(y'_i)$. If $d(y_i) + d(y'_i) \geq n(G) - \sigma(G)$, then we can complete the proof similarly as in Case 1.1. So we assume that $d(y_i) + d(y'_i) < n(G) - \sigma(G)$ for all i, $1 \leq i \leq k$.

Let x_i and x_j be two vertices of B such that $x_i x_j \in E(G)$. Then the subgraph induced by $\{x_j, y_j, y'_j, x_i\}$ is a claw. Since $\eta_{K_{1,3}}(G) \geq n(G) - \sigma(G)$, $d(y_j) + d(y'_j) < n(G) - \sigma(G)$ and $d(y_j) \geq d(y'_j)$, we get that $d(x_i) + d(y_j) \geq n(G) - \sigma(G)$. By Lemma 2, $d(x_i) + d(y_j) + \sigma'(G) \geq n(G)$.

For every end block of G other than C_i, C'_i, C_j, we choose an inner vertex of it as the representative of this end block; and for every non-separable component of G we choose a vertex of it as the representative of this non-separable component. Let S and T be the set of all these representatives of end blocks and non-separable components, respectively. Thus $\mathrm{ebl}(G) = |S| + 3$ and $\mathrm{nco}(G) = |T|$. Let U_i be the union of blocks containing x_i, and let $D_i = U_i - x_i - C_i - C'_i$. Then $d(x_i) \leq n(C_i) + n(C'_i) + |D_i|$ and $d(y_j) \leq n(C_j)$. Note that H is a separable component, and x_i and x_j are chop vertices of G, so $\mathrm{sco}(G) \geq 1$ and $\mathrm{cve}(G) \geq 2$. Thus $d(y) + d(x') + \sigma'(G) \leq n(C_i) + n(C'_i) + |D_i| + n(C_j) + |S| + 3 + |T| - 1 - 1 \leq n(C_i) + n(C'_i) + n(C_j) + |D| + |S| + |T| \leq n$.

We conclude that $d(x_i) + d(y_j) + \sigma'(G) = n(G)$. Note that this equation holds only if x_i is adjacent to every vertex in $C_i \cup C'_i \cup D_i$, y_j is adjacent to every vertex in $C_j - y_j$, every end block other than C_i, C'_i and C_j and every non-separable component of G are trivial, there is only one separable component H, and every vertex of G is either an isolated vertex, a leaf or adjacent to x_i or y_j. By symmetry, we get that for every i, $1 \le i \le k$, x_i is adjacent to every vertex in $C_i \cup C'_i \cup D_i$, y_i is adjacent to every vertex in $C_i - y_i$, and every end block other than C_i, $1 \le i \le k$, and every component of G other than H are trivial. Moreover, this implies that $D = B$ and B is a clique.

Note that H consists of B and C_i, C'_i, $1 \le i \le k$. Let $T = \{z_1, z_2, \ldots, z_t\}$. By Lemma 2, $\sigma(G) \ge k + t$. We claim that $\sigma(G) = k + t$.

Let G' be a graph such that $V(G') = V(G)$ and $E(G') = E(G) \cup \{y'_i y_{i+1} : 1 \le i \le k-1\} \cup \{y'_k z_1\} \cup \{z_j z_{j+1} : 1 \le j \le t\} \cup \{z_t y_1\}$. Then G' is a 2-connected supergraph of G and $|E(G')| - |E(G)| = k + t$. This implies that $\sigma(G) = k + t$.

Similarly as in Case 1.1, we can prove that for every i, $1 \le i \le k$, $C_i \cup \{x_i\}$ contains a Hamilton path starting from x_i.

Let Q'_i be the Hamilton path of $C_i \cup \{x_i\}$ starting from x_i and let $Q_i = y'_i x_i Q'_i$. Then Q_i, $1 \le i \le k$, and all the isolated vertices of T form a path partition of G. Hence $\pi(G) \le k + t = \sigma(G)$. So G is path partition optimal.

Case 2. G has no chop vertices.

We distinguish two subcases: some critical pair is not contained in a block or all critical pairs are contained in a block.

Case 2.1. There is a critical pair that is not contained in a block.

Let $\{x, x'\}$ be a critical pair such that x and x' are not contained in the same block. We treat the two subcases that x and x' are in a common component or in different components, differently.

Case 2.1.1. x and x' are in a common component.

Let H be the component containing x and x'. Note that x and x' are not chop vertices, so each of them is contained in at most two blocks. If x is contained in two end blocks, then H will consist of the two end blocks, and x and x' will be in a common block, a contradiction. Thus we assume that x, and similarly, x', are contained in at most one end block.

Let U be the union of blocks containing x and let U' be the union of blocks containing x'. Then U and U' have at most one common vertex.

For every end block of G not containing x and x', we choose an inner vertex of it as the representative of this end block, and for every non-separable component of G we choose a vertex of it as the representative of this non-separable component. Let S and T be the set of all these representatives of end blocks and non-separable components, respectively. Then $\mathrm{ebl}(G) \le |S|+2$ and $\mathrm{nco}(G) = |T|$. Since H is a separable component, $\mathrm{sco}(G) \ge 1$. Thus $d(x)+d(x')+\sigma'(G) \le n(U-x)+n(U'-x')+|S|+2+|T|-1 \le n(U \cup U')+|S|+|T| \le n(G)$.

Now $d(x)+d(x')+\sigma'(G) = n(G)$. Note that this equation holds only if x is adjacent to every vertex of $U-x$, x' is adjacent to every vertex of $U'-x'$, U and U' both contain exactly one end block, U and U' have exactly one common vertex, every end block of G not containing x and x' and every component of G other than H is trivial, and every vertex of G is either an isolated vertex, a leaf or adjacent to x or x'. This implies that $\sigma(G) = \sigma'(G) = |S|+|T|+1$.

Let y be a neighbor of x in an end block, let y' be a neighbor of x' in an end block, let $S = \{y_1, y_2, \ldots, y_s\}$ and $T = \{z_1, z_2, \ldots, z_t\}$.

We claim that $s = 0$ or 1. Suppose that $s \ge 2$. Let G' be a graph such that $V(G') = V(G)$ and $E(G') = E(G) \cup \{y_i y_{i+1} : 1 \le i \le s-1\} \cup \{yz_1\} \cup \{z_j z_{j+1} : 1 \le j \le t-1\} \cup \{z_t y'\}$. Then G' is a 2-connected supergraph of G and $|E(G')| - |E(G)| = s+t$. This implies that $\sigma(G) \le s+t$, a contradiction.

If $s = 0$, then H is a block-chain. We claim that H is $\{K_{1,3}, N\}$-o_{-1}-heavy. Let $\{u, v\}$ be a critical pair of G contained in H. Then $d_H(u)+d_H(v) = d(u)+d(v) \ge n(G)-\sigma(G) = n(G)-t-1 = n(H)-1$. Since $\eta_{K_{1,3}}(G) \ge n(G)-\sigma(G)$ and $\eta_N(G) \ge n(G)-\sigma(G)$, we conclude that H is $\{K_{1,3}, N\}$-o_{-1}-heavy. By Theorem 9.12, H is traceable. Let Q be a Hamilton path of H. Then Q and all the isolated vertex of T form a path partition of G. Thus $\pi(G) \le t+1$ and G is path partition optimal.

Now we assume that S has exactly one vertex, say y''. Let z be the (only) common vertex of U and U', let C and C' be the two components of $H-z$ containing x and x', respectively, and let $B = C \cup \{z\}$ and $B' = C' \cup \{z\}$. Without loss of generality, we assume that x is a critical vertex.

Note that B is a block-chain. We claim that B is $\{K_{1,3}, N\}$-o_{-1}-heavy. Let $\{u, v\}$ be a critical pair of G such that $u, v \in V(B)$. Without loss of generality, we assume that $u \neq z$. Then $d_B(u) = d(u)$, $d_B(v) \geq d(v) - n(C' - y')$ and $d_B(u) + d_B(v) \geq d(u) + d(v) - n(C') + 1 \geq n(G) - \sigma(G) - n(C') + 1 = n(G) - (t+2) - n(C') + 1 = n(B) - 1$. This implies that B is $\{K_{1,3}, N\}$-o_{-1}-heavy. By Theorem 9.12, B contains a Hamilton path.

Now we claim that C' contains a Hamilton path. Let D' be the end block containing x'. Note that x' is adjacent to every vertex in $C' - \{x', y''\}$. If there are two nonadjacent vertices in $D' - x'$ or in $C' - D' - y''$, then there will be an induced claw in C' with center x', and there will be a critical vertex x'' in C' other than x'. Thus $\{x, x''\}$ is a critical pair such that x and x'' are not in a common block. By the above analysis, x'' is contained in an end block and x'' is adjacent to every vertex in $C' - (D' - x')$. This implies that x'' is the (unique) neighbor of y''. Using the same analysis, we can deduce that D' is trivial. This will cause that x'' is nonadjacent to some vertex in $C' - (D' - x')$, a contradiction. So we assume that D' and $C' - D' - y''$ are cliques. Then C' is clearly traceable.

Let Q be a Hamilton path of B, and let Q' be a Hamilton path of C'. Then Q, Q', and all the isolated vertices of T form a path partition of G. Hence $\pi(G) \leq t + 2 = \sigma(G)$. So G is path partition optimal.

Case 2.1.2. x and x' are in distinct components.

Let H be the component containing x, and let H' be the component containing x'. Let U be the union of blocks containing x, let U' be the union of blocks containing x', and let $C = U - x$ and $C' = U' - x'$.

For every end block of G not containing x and x', we choose an inner vertex of it as the representative of this end block, and for every non-separable component of G other that H and H', we choose a vertex of it as the representative of this non-separable component. Let S and T be the set of all these representatives of end blocks and non-separable components.

If x is contained in one or two end blocks, then H is a separable component; if x' is contained in one or two end blocks, then H' is a separable component.

If x and x' are both contained in no end blocks, then $U \cap S = \emptyset$ and $U' \cap S = \emptyset$. Note that $\mathrm{ebl}(G) = |S|$, $\mathrm{nco}(G) \leq |T| + 2$ and $\mathrm{sco}(G) \geq 0$.

We get $d(x) + d(x') + \sigma'(G) \leq n(U - x) + n(U' - x') + |S| + |T| + 2 \leq n(U) + n(U') + |S| + |T| \leq n(G)$.

If x is contained in no end blocks and x' is contained in at least one end block, then $U \cap S = \emptyset$. Note that $\text{ebl}(G) \leq |S| + 2$, $\text{nco}(G) \leq |T| + 1$ and $\text{sco}(G) \geq 1$. We get $d(x) + d(x') + \sigma'(G) \leq n(U - x) + n(U' - x') + |S| + 2 + |T| + 1 - 1 \leq n(U) + n(U') + |S| + |T| \leq n(G)$.

If x is contained in at least one end block and x' is contained in no end blocks, then we can similarly obtain that $d(x) + d(x') + \sigma'(G) \leq n(G)$.

If x and x' are both contained in at least one end block, then $\text{ebl}(G) \leq |S| + 4$, $\text{nco}(G) = |T|$ and $\text{sco}(G) \geq 2$. We get $d(x) + d(x') + \sigma'(G) \leq n(U - x) + n(U' - x') + |S| + 4 + |T| - 2 \leq n(U) + n(U') + |S| + |T| \leq n(G)$.

Thus in any case, $d(x) + d(x') + \sigma'(G) = n(G)$. Note that this equation holds only if x is adjacent to every vertex of $U - x$, x' is adjacent to every vertex of $U' - x'$, H is either a non-separable component or consists of two end blocks, H' is either a non-separable components or consist of two end blocks, every component of G other than H and H' is trivial, and every vertex of G is either an isolated vertex, a leaf or adjacent to x or x'. This implies that $\sigma(G) = \sigma'(G) = |T| + 2$. Let $t = |T|$.

Note that H is a block-chain. We claim that H is $\{K_{1,3}, N\}$-o_{-1}-heavy. Let $\{u, v\}$ be a critical pair of G contained in H. Then $d_H(u) + d_H(v) = d(u) + d(v) \geq n(G) - \sigma(G) = n(G) - t - 2 \geq n(H) - 1$. Since $\eta_{K_{1,3}}(G) \geq n(G) - \sigma(G)$ and $\eta_N(G) \geq n(G) - \sigma(G)$. So H is $\{K_{1,3}, N\}$-o_{-1}-heavy. By Theorem 9.12, H is traceable. Similarly, we can prove that H' is traceable.

Let Q be a Hamilton path of H, and let Q' be a Hamilton path of H'. Then Q, Q', and all the isolated vertices of T form a path partition of G. Thus we have $\pi(G) \leq t + 2 = \sigma(G)$. So G is path partition optimal.

Case 2.2. Every critical pair is contained in a block.

By our assumption, there is an induced copy of $K_{1,3}$ or N in G. Thus there is at least one critical pair and at least one critical vertex. We assume here that every critical pair is contained in a block. It is not difficult to see that this implies that all the critical pairs are contained in a common block. Let B be a block containing all the critical pairs, and let H be the component containing B.

Let x_i, $1 \leq i \leq k$, be the cut vertices contained in B. For every i, $1 \leq i \leq k$, Let x'_i be a neighbor of x_i in B, let y_i be a neighbor of x_i in $H - B$, let C_i be the component of $H - x_i$ containing y_i, and let D_i be the subgraph of G induced by $C_i \cup \{x_i, x'_i\}$.

Since D_i contains no critical pairs, D_i is $\{K_{1,3}, N\}$-free, and thus is traceable by Theorem 9.3. This implies that D_i is a block-chain and $C_i \cup \{x_i\}$ contains exactly one end block of G. Let y'_i be an inner vertex of the end block contained in $C_i \cup \{x_i\}$.

Let H_j, $1 \leq j \leq t$, be the components of G other than H. For every j, $1 \leq j \leq t$, since H_j contains no critical pairs, H_j is $\{K_{1,3}, N\}$-free, and thus is traceable by Theorem 9.3. This implies that H_j is a block-chain. If H_j is trivial, then let $z_j = z'_j$ be the (only) vertex in H_j; if H_j is non-trivial and non-separable, then let z_j and z'_j be two distinct vertices of H_j; and if H_j is separable, then let z_j and z'_j be inner vertices of two distinct end blocks of H_j.

We first assume that H is non-separable. Then $k = 0$ and $t \geq 1$. By Lemma 2, $\sigma(G) \geq t + 1$. We claim that $\sigma(G) = t + 1$. Let y, y' be two distinct vertices of H. Let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G) \cup \{yz_1\} \cup \{z'_j z_{j+1} : 1 \leq j \leq t - 1\} \cup \{z'_t y'\}$. Then G' is a 2-connected supergraph of G and $|E(G')| - |E(G)| = t+1$. This implies that $\sigma(G) = t+1$.

Note that H is a block-chain. We claim that H is $\{K_{1,3}, N\}$-o_{-1}-heavy. Let $\{u, v\}$ be a critical pair of G contained in H. Then $d_H(u)+d_H(v) = d(u)+d(v) \geq n(G) - \sigma(G) = n(G) - t - 1 \geq n(H) - 1$. Since $\eta_{K_{1,3}}(G) \geq n(G) - \sigma(G)$ and $\eta_N(G) \geq n(G) - \sigma(G)$, we conclude that H is $\{K_{1,3}, N\}$-o_{-1}-heavy. By Theorem 9.12, H is traceable.

Recall that every component of G other than H is also traceable. All the Hamilton paths of the components of G form a path partition of G. This implies that $\pi(G) \leq t + 1 = \sigma(G)$, so G is path partition optimal.

Now we assume that $k \geq 1$. By Lemma 2, $\sigma(G) \geq \lceil k/2 \rceil + t$. We claim that $\sigma(G) = \lceil k/2 \rceil + t$. If k is even, then let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G) \cup \{y'_{2i-1} y'_{2i} : 1 \leq i \leq (k-2)/2\} \cup \{y'_{k-1} z_1\} \cup \{z'_j z_{j+1} : 1 \leq j \leq t-1\} \cup \{z'_t y'_k\}$. If k is odd, then let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G) \cup \{y'_{2i-1} y'_{2i} : 1 \leq i \leq (k-1)/2\} \cup \{y'_k z_1\} \cup \{z'_j z_{j+1} : 1 \leq j \leq t-1\} \cup \{z'_t y'_1\}$. Then G' is a 2-connected supergraph of G and $|E(G'')| - |E(G)| = \lceil k/2 \rceil + t$. This implies that $\sigma(G) = \lceil k/2 \rceil + t$.

If $t \geq 1$, then let G' be the graph obtained from G by deleting the component H_t. Clearly, $\eta_{K_{1,3}}(G') \geq n(G') - \sigma(G')$ and $\eta_N(G') \geq n(G') - \sigma(G')$. By the induction hypothesis, $\pi(G') = \sigma(G') = \lceil k/2 \rceil + t - 1$. Note that a path partition of G' together with a Hamilton path of H_t forms a path partition of G. This implies that $\pi(G) \leq \lceil k/2 \rceil + t$, so G is path partition optimal.

Next we assume that $t = 0$, implying that there is only one component H of G. If there is some i, $1 \leq i \leq k$, such that C_i is non-trivial, then let G' be the graph obtained from G by deleting $C_i - z_i$. Clearly, $\eta_{K_{1,3}}(G') \geq n(G') - \sigma(G')$ and $\eta_N(G') \geq n(G') - \sigma(G')$. By the induction hypothesis, $\pi(G') = \sigma(G') = \lceil k/2 \rceil$. Let $\mathcal{P}'$ be a path partition of G' with $\lceil k/2 \rceil$ paths, and let P_i be a Hamilton path of D_i. If y_i is a trivial path in $\mathcal{P}'$, then let $Q_i = P_i - \{x_i, y_i\}$, and $\mathcal{P} = \mathcal{P}' \setminus \{y_i\} \cup \{Q_i\}$ is a path partition of G. If z_i is contained in some non-trivial path Q_i', then Q_i' contains the edge $x_i z_i$. Let $Q_i = (Q_i' - x_i y_i) \cup (P_i - x_i x_i')$, and $\mathcal{P} = \mathcal{P}' \setminus \{Q_i'\} \cup \{Q_i\}$ is a path partition of G. Hence $\pi(G) \leq \lceil k/2 \rceil$, so G is path partition optimal.

Next we assume that for every i, $1 \leq i \leq k$, C_i is trivial. If $k \geq 3$, then let G' be the graph with $V(G') = V(G) \setminus \{z_{k-1}, z_k\} \cup \{z\}$ and $E(G') = E(G - \{z_{k-1}, z_k\}) \cup \{x_{k-1}z, x_k z\}$. Note that $d_{G'}(z) = 2$ and any copy of $K_{1,3}$ or N in G' contains at most one edge in $\{x_{k-1}z, x_k z\}$. Every induced copy of $K_{1,3}$ or N in G' is also induced in G. Let $\{u, v\}$ be a critical pair of G. Then $u, v \in B$, $d_{G'}(u) = d(u)$, and $d_{G'}(v) = d(v)$. Noting that $n(G') = n(G) - 1$ and $\sigma(G') = \sigma(G) - 1$, we see that $\eta_{K_{1,3}}(G') \geq n(G') - \sigma(G')$ and $\eta_N(G') \geq n(G') - \sigma(G')$. By the induction hypothesis, $\pi(G') = \sigma(G') = \lceil k/2 \rceil - 1$. Let $\mathcal{P}'$ be a path partition of G' with $\lceil k/2 \rceil - 1$ paths, and let Q be the path in $\mathcal{P}'$ containing z. Let P and P' be the subpaths of Q ending at and starting from z such that $Q = PzP'$. Then $\mathcal{P} = \mathcal{P}' \setminus \{Q\} \cup \{P, P'\}$ is a path partition of G. This implies that $\pi(G) \leq \lceil k/2 \rceil$, so G is path partition optimal.

Finally, we assume that $k = 1$ or 2. Then G is a block-chain. By Theorem 9.12, G is path partition optimal.

This completes the proof of Theorem 9.10.

Summary

The research that forms the basis of this thesis addresses the following general structural questions in graph theory: which fixed graph or pair of graphs do we have to forbid as an induced subgraph of an arbitrary graph G to guarantee that G has a nice structure? In this thesis the nice structural property we have been aiming for is the existence of a Hamilton cycle, i.e., a cycle containing all the vertices of the graph, or related properties like the existence of a Hamilton path, of cycles of every length, or of Hamilton paths starting at every vertex of the graph. For these structural properties, sufficient Ore-type degree conditions (where the degree of a vertex is the number of neighbors of that vertex, or equivalently, the number of edges incident with that vertex) are known since the 1960s. These Ore-conditions are of the type: if every pair of nonadjacent vertices of the graph G has degree sum at least some lower bound, then G is guaranteed to have the structural property. For the existence of a Hamilton cycle the critical lower bound is the number of vertices of the graph, for a Hamilton path it is the number of vertices minus one, and for cycles of every length it is the number of vertices plus one. In order to obtain common generalizations of these sufficiency results based on Ore-type degree sum conditions on one hand and forbidden induced subgraph conditions on the other hand, the following questions have also been addressed in the thesis. Can we restrict the corresponding Ore-type degree sum condition to certain induced subgraphs or pairs of induced subgraphs of a graph G and still guarantee that G has the same nice structure? In the thesis work we have proved many examples that provide affirmative answers to these general questions. For convenience, we will not go into the details and subtle differences of the

definitions for the different concepts, but say that an induced subgraph H of a graph G is heavy for some property if there is a pair of nonadjacent vertices of H with degree sum at least the Ore-type degree lower bound of G for that property. In that case we say that G is H-heavy if every induced subgraph of G isomorphic to H is heavy (for that property). We refer to the listed chapters for the details and for the precise definitions and formulations of the results.

Chapter 1 contains a short general introduction to the topics of the thesis and gives an overview of the main results, together with some motivation and connections to and relationships with older results. Specific terminology and notation can be found just before each of the topics.

In Chapter 2, we first characterize all the graphs that do not contain a heavy cycle (a cycle of a graph G containing all the vertices with degree at least $|V(G)|/2$). We use this result to characterize all the connected graphs S with the following property: every longest cycle of a 2-connected S-free (or S-heavy) graph is a heavy cycle.

In Chapter 3, we characterize the pairs of connected graphs R, S such that every connected R-heavy and S-heavy graph is traceable, i.e., contains a Hamilton path. We also determine the graphs S such that every connected $K_{1,3}$-heavy and S-free graph is traceable.

In Chapter 4, we consider forbidden subgraph conditions for block-chains (graphs whose block-tree is a path) to be traceable. We characterize the pairs of connected graphs R, S such that every R-free and S-free block-chain is traceable.

In Chapter 5, we consider heavy subgraph conditions for traceability of block-chains. We characterize the pairs of connected graphs R, S such that every R-heavy and S-heavy block-chain is traceable.

In Chapter 6, we consider heavy subgraph conditions for hamiltonicity. We characterize the pairs of graphs R, S such that every 2-connected R-heavy and S-heavy graph is hamiltonian.

In Chapter 7, we consider forbidden subgraph conditions for a 2-connected graph to be homogeneously traceable, i.e., such that there is a Hamilton path

starting at every vertex. We characterize the pairs of graphs R, S such that every 2-connected R-free and S-free graph is homogeneously traceable. We also characterize the heavy subgraph pairs for this property.

In Chapter 8, we consider heavy subgraph conditions for pancyclicity, i.e., for the existence of cycles of every length between 3 and the number of vertices of the graph. We characterize the pairs of graphs R, S such that every 2-connected R-heavy and S-heavy graph that is not a cycle is pancyclic.

In Chapter 9, we introduce and deal with the path partition number and separable degree of graphs. We give forbidden and heavy subgraph conditions for a graph to have path partition number equal to the separable degree, thereby extending existing results on hamiltonicity.

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Acknowledgements

Working towards a PhD thesis is a long road with many bends and turns. It is impossible to do on one's own. I obtained great help and useful advice from a lot of people. It was this support that kept me going on this road, and made this trip into one of the most important trips of my life. I am grateful to all who supported me on this journey one way or the other, including some people which I may forget to mention here, although not intentionally.

First of all, I would like to thank my supervisors Professor Dr. Shenggui Zhang and Professor Dr. Ir. Hajo Broersma. I thank S. Zhang in particular for his optimism and for his encouraging and positive supervision. I am grateful to H.J. Broersma for inviting me to be a PhD student at the University of Twente and for his useful comments during meetings. This thesis could not have been written without his stimulation, instruction and cooperation. His profound knowledge of graph theory and scrupulous scientific spirit have been very helpful in the past and will be useful too in my further research.

I enjoyed the pleasant working environment at the group of Discrete Mathematics and Mathematical Programming at the University of Twente. I want to thank everyone from DMMP for this, and in addition I am grateful to Marc Uetz, Georg Still and Marjo Mulder for their help. I greatly appreciate Dongshuang Hou's help for my personal life in the Netherlands. Thanks also go to my colleagues and friends from Northwestern Polytechnical University, namely Ying Wang, Bo Ning, Yandong Bai and Bolin Chen, for advising me well to go ahead despite difficulties.

A lot of thanks go to the friends I met in Enschede: Yuan Feng, Xian Qiu, Lulu Tian, Huan Yang, Haishang Cao, Hongxu Guo, Chao Xia, Zhao Wang, Changxin Liu, Zhipei Fu and Ying Du. You are great friends for bringing me lots of laughter and happiness.

Last but not least, I want to express my deep thanks to my parents, my brother and my sister. They provided me with generous support and encouragement during these years, even though they did not always understand what I was doing.

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August 2012, Enschede

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Binlong Li was born on August 2, 1983, in Pingshan County of Hebei Province, P.R. China. From 1991 until 2003 he attended primary and secondary school in his hometown. In September 2003, he started to study at Northwestern Polytechnical University in Xi'an. After receiving his Bachelor degree, he studied mathematics at the same university and became a graduate student. He specialized in operations research, in particular graph theory. He graduated with honours and received his Master degree after completing a Master's thesis, entitled 'Heavy Paths and Cycles in Weighted Graphs and Weighted Digraphs', under the supervision of Professor Dr. Shenggui Zhang.

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